p-operator spaces and harmonic analysis: Feichtinger–Figà-Talamanca–Herz algebras

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Feichtinger's minimal Segal algebra

Definition [Reiter '60s]

Let G be a l.c.g. A subspace $\mathrm{S}^1(G) \subset \mathrm{L}^1(G)$ is a <u>Segal algebra</u> if it is dense and has a left translation invariant norm, continuous in G, $\|\cdot\|_{\mathrm{S}^1} \geq \|\cdot\|_{\mathrm{L}^1}$ by which it is complete.

Theorem [Feichtinger '80]

For every l.c. abelian g. G, there exists $\mathrm{S}_0(G) \subset \mathrm{L}^1(G)$ s.t.

(a) $\mathrm{S}_0(G)$ is minimal Segal algebra for which $\chi \mathrm{S}_0(G) \subset \mathrm{S}_0(G)$ (pointwise mult'n) with isometric action, for χ in $\widehat{G}$;

(b) $\widehat{\mathrm{S}_0(G)} = \mathrm{S}_0(\widehat{G})$ (Fourier transform);

(c) $H \leq G$ closed, $\mathrm{S}_0(G)|_H = \mathrm{S}_0(H)$ and $T_H(\mathrm{S}_0(G)) \cong \mathrm{S}_0(G/H)$ (T_H averaging over H); and

(d) $\mathrm{S}_0(G) \otimes^\gamma \mathrm{S}_0(H) \cong \mathrm{S}_0(G \times H)$ (H l.c.a.g., $\otimes^\gamma$ proj've t.p.).

Definition [Leinert, Burnham '70s]

Let $\mathcal{A}$ be a Banach algebra. An abstract Segal algebra is dense ideal $\mathcal{S} \subset \mathcal{A}$, which is complete w.r.t. a norm $\|\cdot\|_{\mathcal{S}} \geq \|\cdot\|_{\mathcal{A}}$ by which it is a left Banach $\mathcal{A}$-module.

Adding operator space structure ...

Definition [after Forrest-S.-Wood '07]

Let $\mathcal{A}$ be completely contractive Banach algebra. An operator Segal algebra is dense ideal $\mathcal{S} \subset \mathcal{A}$, which admits a complete operator space structure w.r.t. which $\mathcal{S} \hookrightarrow \mathcal{A}$ completely boundedly, and $\mathcal{S}$ is a completley bounded left $\mathcal{A}$-module.

Operator projective tensor product

[Effros-Ruan '90] $\mathcal{M}, \mathcal{N}$ v.N. alg's, $(\mathcal{M}\overline{\otimes}\mathcal{N})_* \cong \mathcal{M}_*\hat{\otimes}\mathcal{N}_*$

$\Rightarrow$ Fourier algebras $\mathrm{A}(G \times H) \cong \mathrm{A}(G)\hat{\otimes}\mathrm{A}(H)$

[Losert '84] $\mathrm{A}(G \times H) \cong \mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(H) \Leftrightarrow$ one of G, H a.a.

Extending Feichtinger's algebra to non-abelian groups

G abelian, $\mathrm{A}(G) = \widehat{\mathrm{L}^1(\hat{G})}$ (inv. Fourier trans.); gen. [Eymard '64]

Theorem [S. '07]

For every l.c.g. G, $\exists$ an operator Segal algebra $\mathrm{S}_0(G) \subset \mathrm{A}(G)$ s.t.

(a) $\mathrm{S}_0(G)$ is the minimal Segal algebra in $\mathrm{A}(G)$ which is closed under left translations, and for which such translations are (complete) isometries and continuous in G;

(b) $\mathrm{S}_0(G) \subset \mathrm{L}^1(G)$, and is a(n operator) Segal ideal therein;

(c) $H \leq G$ closed, $\mathrm{S}_0(G)|_H = \mathrm{S}_0(H)$ (c'ly) surjectively;

(d) $N \lhd G$ closed, $T_N(\mathrm{S}_0(G)) \cong \mathrm{S}_0(G/N)$ (c'ly) surjectively; and

(e) $\mathrm{S}_0(G)\hat{\otimes}\mathrm{S}_0(H) \cong \mathrm{S}_0(G \times H)$ (c'ly) isomorphically.

Theorem [Öztop-S.]

$\mathrm{S}_0(G)$ is the minimal Segal algebra in $\mathrm{L}^1(G)$ for which $\mathrm{A}(G)\mathrm{S}_0(G) \subset \mathrm{S}_0(G)$ (pointwise mult'n).

Figà-Talamanca–Herz algebras

$1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, G – l.c.g.

$\lambda_p : G \to \mathcal{B}(\mathrm{L}^p(G))$ – p-left reg. rep'n, $\lambda_p(s)\eta(t) = \eta(s^{-1}t)$

$\mathrm{N}^p(G) = \mathrm{L}^{p'}(G) \otimes^\gamma \mathrm{L}^p(G), \quad \mathrm{N}^p(G)^* \cong \mathcal{B}(\mathrm{L}^p(G))$

$P_G : \mathrm{N}^p(G) \to \mathcal{C}_0(G), \qquad P_G \xi \otimes \eta = (s \mapsto \langle \xi, \lambda_p(s)\eta \rangle)$

$$\mathrm{A}_p(G) = \mathrm{ran} P_G \text{ (quotient space of } \mathrm{N}^p(G))$$

$$\mathrm{PM}_p(G) = (\ker P_G)^\perp \subset \mathcal{B}(\mathrm{L}^p(G)), \quad \mathrm{PM}_p(G) \cong \mathrm{A}_p(G)^*$$

Bipolar theorem $\Rightarrow \mathrm{PM}_p(G) = \overline{\mathrm{span}}^{w*} \lambda_p(G)$.

Figà-Talamanca–Herz algebras

Proposition [Daws '10, after Herz '71]

$\mathrm{PM}_p(G)\bar{\otimes}\mathrm{PM}_p(G) = \overline{\mathrm{span}}^{w*}\lambda_p(G)\otimes\lambda_p(G) \cong \mathrm{PM}_p(G\times G)$

2nd idf'n spatial: $\mathrm{L}^p(G)\otimes^p \mathrm{L}^p(G) \cong \mathrm{L}^p(G\times G)$.

Proposition [Daws '10] ([Stinespring '59], [Herz '70])

$W_G \in \mathcal{B}(\mathrm{L}^p(G)\otimes\mathrm{L}^p(G))$, $W_G\xi(s,t) = \xi(s,st)$ satisfies

$$W_G(\lambda(s)\otimes I)W_G^{-1} = \lambda_p(s)\otimes\lambda_p(s)$$

$\Rightarrow$ pointwise mult'n $\mathrm{A}_p(G)\otimes^\gamma \mathrm{A}_p(G)\to\mathrm{A}_p(G)$ defined and cts.

Theorem [Herz '73]

The Gelfand spectrum of $\mathrm{A}_p(G)$ is G.

p-operator spaces (Pisier, LeMerdy, after Ruan)

$\mathcal{V}$ – vector space, p-op. space structure $\{\|\cdot\|_n : \mathrm{M}_n(\mathcal{V}) \to \mathbb{R}^{\geq 0}\}$ s.t.

$$(D_\infty) \qquad \left\| \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \right\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$$

$$(M_p) \qquad \|\alpha v \beta\|_n \leq \|\alpha\|_{\mathcal{B}(\ell^p_n)} \|v\|_n \|\beta\|_{\mathcal{B}(\ell^p_n)}, \alpha, \beta \in \mathrm{M}_n$$

$S : \mathcal{V} \to \mathcal{W}$, $S^{(n)} : \mathrm{M}_n(\mathcal{V}) \to \mathrm{M}_n(\mathcal{W})$, $S^{(n)}[v_{ij}] = [Sv_{ij}]$
S completely bounded if $\|S\|_{p\mathrm{cb}} = \sup_n \left\|S^{(n)}\right\|$,
c'ly contractive if $\|S\|_{p\mathrm{cb}} \leq 1$, c'te quotient if each $S^{(n)}$ quotient

Theorem (Pisier, LeMerdy; Ruan if $p = 2$)

$\mathcal{V}$ p-operator space, $\exists$ complete isometry $\pi : \mathcal{V} \to \mathcal{B}(E)$, $E \in \mathcal{SQ}_p$

Here $\mathrm{M}_n(\mathcal{B}(E)) = \mathcal{B}(\ell^p(n, E))$.
Danger. No known Wittstock–Haagerup–Paulsen extension theorem, i.e. $\mathrm{M}_n = \mathcal{B}(\ell^p_n)$ not known to be injective, in any sense.

Mapping and dual spaces ([Blecher, Effros–Ruan '90s])

$\mathrm{M}_n(\mathcal{CB}_p(\mathcal{V},\mathcal{W})) = \mathcal{CB}_p(\mathcal{V},\mathrm{M}_n(\mathcal{W}))$ satisfies (D_∞) and (M_p)

Proposition [Daws '10]

$\mathcal{V}^* = \mathcal{CB}_p(\mathcal{V},\mathbb{C})$ isometrically; $\mathrm{A}_p(G)\tilde{\subset}\mathrm{PM}_p(G)^*$

Hence $\mathrm{M}_n(\mathcal{V}^*) = \mathcal{CB}_p(\mathcal{V},\mathcal{B}(\ell^p_n))$.

Proposition [Daws '10]

(a) $\mathcal{V}^* \hookrightarrow \mathcal{B}(\ell^p(I))$ completely isometrically for some I

(b) $S:\mathcal{V}\to\mathcal{W}$ complete contraction $\Rightarrow S^*:\mathcal{W}^*\to\mathcal{V}^*$ complete contraction

(c) $\kappa_\mathcal{V}:\mathcal{V}\to\mathcal{V}^{**}$ complete contraction; complete isometry $\Leftrightarrow \mathcal{V}\hookrightarrow\mathcal{B}(\mathrm{L}^p(\mu))$ completley isometrically for some μ

$\mathcal{V}$ <u>acts on L^p</u> if $\mathcal{V}\hookrightarrow\mathcal{B}(\mathrm{L}^p(\mu))$ completely isometrically for some μ

Tensor products and weakly completley bounded maps

$\mathcal{V}\hat{\otimes}^p\mathcal{W} = \overline{\mathcal{V}\otimes\mathcal{W}}^{\|\cdot\|_{\wedge p}}$ – p-operator projective tensor product

Proposition [Daws '10] ([Blecher–Paulsen, Effros–Ruan '90s])

(a) $(\mathcal{V}\hat{\otimes}^p\mathcal{W})^* = \mathcal{CB}_p(\mathcal{V},\mathcal{W}^*)$ completely isometrically

(b) commutative, projective

Definition

$S:\mathcal{V}\to\mathcal{W}$ weakly completely bounded (resp. contractive, quotient) if $S^*:\mathcal{W}^*\to\mathcal{V}^*$ is c.b. (resp. c.c., c.i.)

Proposition

(a) [Daws '10] $\mathrm{N}^p(\mu)\hat{\otimes}^p\mathrm{N}^p(\nu)\cong\mathrm{N}^p(\mu\times\nu)$, w.c.i'ly, $\forall\ \mu,\nu$

(b) $S:\mathcal{V}\to\mathcal{W}$ w.c.b. (q.)
$\Leftrightarrow S\otimes\mathrm{id}:\mathcal{V}\hat{\otimes}^p\mathrm{N}^p(\mu)\to\mathcal{W}\hat{\otimes}^p\mathrm{N}^p(\mu)$ b. (q.) $\forall\ \mu$

(c) $S:\mathcal{V}\to\mathcal{W}$ w.c.b. & $\mathcal{W}$ acts on L^p $\Rightarrow$ S c.b.

Tensor products of Figà-Talamanca–Herz algebras

Theorem (after [Daws '10])

(a) $u \otimes v \mapsto u \times v : \mathrm{A}_p(G)\hat{\otimes}^p \mathrm{A}_p(H) \to \mathrm{A}_p(G \times H)$ is w.c.q.

(b) $\mathrm{A}_p(G)\hat{\otimes}^p \mathrm{A}_p(H) = \mathrm{A}_p(G \times H)$ w.c.i'ly, if G, H amenable

Proof of (b) uses that

$$(\mathrm{A}_p(G)\hat{\otimes}^p \mathrm{A}_p(H))^* \cong \mathrm{PM}_p(G)\bar{\otimes}_F \mathrm{PM}_p(H) \tilde{\subset} \mathrm{CV}_p(G \times H).$$

[Cowling '98] [Daws-S. '13] $\mathrm{PM}_p(G) = \mathrm{CV}_p(G)$ if G has AP
$\Rightarrow$ (b), above, holds for G, H with AP.

Conjecture: $p \neq 2$, G, H infinite, $\mathrm{A}_p(G) \otimes^\gamma \mathrm{A}_p(H) \neq \mathrm{A}_p(G \times H)$.

Proposition (after [Daws '10])

mult : $\mathrm{A}_p(G)\hat{\otimes}^p \mathrm{A}_p(G) \to \mathrm{A}_p(G)$ c.c. & w.c.q.; $\mathrm{A}_p(G)$ c.c.B.a.

A special class of ideals

$K \subset G$ compact, non-null
$\mathcal{M}_p(K) = P_G(\mathrm{N}^p(K))$ where $\mathrm{N}^p(K) = \mathrm{L}^p(K) \otimes^\gamma \mathrm{L}^p(K) \subset \mathrm{N}^p(G)$
with quotient o.s. struc. $\rightsquigarrow$ dual o.s.s. from $\mathcal{M}_p(K) \hookrightarrow \mathcal{M}_p(K)^{**}$
(Related: [Cowling '98].)

Theorem

$\mathcal{M}_p(K)$ operator Segal ideal in $\mathrm{A}_p(G)$.

Proof uses

- $W_K \in \mathcal{B}(\mathrm{L}^p(K \times G))$, $W_K\eta(s,t) = \eta(s,st)$
- $\mathcal{M}_p(K)^* = \overline{1_K \mathrm{PM}_p(G) 1_K}^{w*}$, then

$$\begin{aligned} 1_K\lambda_p(s)1_K &\mapsto 1_K\lambda_p(s)1_K \otimes I \\ &\mapsto W_K(1_K\lambda_p(s)1_K \otimes I)W_K^{-1} = 1_K\lambda_p(s)1_K \otimes \lambda_p(s). \end{aligned}$$

$\Rightarrow$ mult: $\mathcal{M}_p(K)\hat{\otimes}^p \mathrm{A}_p(G) \to \mathcal{M}_p(K)$ w.c.c.

Construction of $\mathrm{S}_0^p(G)$

$\ell^1(G)\tilde{\subset}\ell^\infty(G)^*$, $\ell^\infty(G)\tilde{\subset}\mathcal{B}(\ell^p(G))$ – min. space
$\Rightarrow$ $\ell^1(G)$ max'l space acting on L^p ([Öztop-S. '12]);
$\ell^1(G)\hat{\otimes}^p\mathcal{V} = \ell^1(G)\otimes^\gamma\mathcal{V} \cong \ell^1(G,\mathcal{V})$

$Q_K : \ell^1(G)\hat{\otimes}^p\mathcal{M}_p(K) \to \mathrm{A}_p(G)$, $Q(\delta_s\otimes u) = s * u$
$\mathrm{S}_0^p(G) = \mathrm{ran}\, Q_K$, quotient o.s. struc. $\rightsquigarrow$ dual o.s. struc.

Theorem

(a) $\mathrm{S}_0^p(G)$ operator Segal algebra in $\mathrm{A}_p(G)$
(b) description independant of compact non-null K
(c) can replace $\mathcal{M}_p(K)$ with other compactly supported op. Segal ideals, say $\mathrm{A}_p^K(G) = \{u \in \mathrm{A}_p(G) : \mathrm{supp}\, u \subset K\}$
(d) minimal Segal algebra in $\mathrm{A}_p(G)$, closed under bdd. cts. translations

Call $\mathrm{S}_0^p(G)$ the *p*-<u>Feichtinger–Figà-Talamanca–Herz algebra</u>.

$\mathrm{S}_0^p(G)$ as a Segal algebra in $\mathrm{L}^1(G)$

weakly compete surjection: $S : \mathcal{V} \to \mathcal{W}$ s.t.
induced $\bar{S} : \mathcal{V}/\ker S \to \mathcal{W}$ w.c. isomorphism
$\Leftrightarrow S \otimes \mathrm{id} : \mathcal{V}\hat{\otimes}\mathrm{N}^p(\mathbb{N}) \to \mathcal{W}\hat{\otimes}\mathrm{N}^p(\mathbb{N})$ surjective

Theorem

(a) $\mathrm{S}_0^p(G)$ pseudo-symmetric operator Segal algebra in $\mathrm{L}^1(G)$

(b) $\mathrm{S}_0^p(G) = \mathrm{ran}\, Q'_K$, $Q'_K : \mathrm{L}^1(G)\hat{\otimes}^p\mathcal{M}_p(K) \to \mathrm{S}_0^p(G)$, $Q'_K(f \otimes u) = f * u$, w.c. surj'n

(c) minimal Segal algebra in $\mathrm{L}^1(G)$, closed under p'twise mult'n by $\mathrm{A}_p(G)$.

Functorial property: tensor products

Theorem

G, H – l.c.gs.

(a) $K \subset G$, $L \subset H$ compact non-null;
$u \otimes v \mapsto u \times v : \mathcal{M}_p(K)\hat{\otimes}^p\mathcal{M}(L) \to \mathcal{M}_p(K \times L)$ is w.c.q.,
injective if $\mathrm{A}_p(G)\hat{\otimes}^p\mathrm{A}_p(H) = \mathrm{A}_p(G \times H)$

(b) $u \otimes v \mapsto u \times v : \mathrm{S}_0^p(G)\hat{\otimes}^p\mathrm{S}_0^p(H) \to \mathrm{S}_0^p(G \times H)$ is w.c. surj.,
injective if $\mathrm{A}_p(G)\hat{\otimes}^p\mathrm{A}_p(H) = \mathrm{A}_p(G \times H)$

This may be considered main reason for introducing p-op. space:

$\mathrm{S}_0^p(G) \otimes^\gamma \mathrm{S}_0^p(H) \cong \mathrm{S}_0^p(G \times H)$ known only if one G, H discrete.

Functorial property: restriction to subgroups

$H \leq G$ closed

Theorem (after [Derighetti '84])

$\exists$ complete isometry $\iota : \mathrm{CV}_p(H) \hookrightarrow \mathrm{CV}_p(G)$ s.t. $\iota|_{\mathrm{PM}_p(H)} = \mathrm{Res}_H{}^*$. Hence $\mathrm{Res}_H : \mathrm{A}_p(G) \to \mathrm{A}_p(H)$ w.c.q.

Theorem

(a) $\mathrm{Res}_H(\mathcal{M}_p(K))$ quot.o.s.s. $\rightsquigarrow$ dual o.s.s., Segal ideal in $\mathrm{A}_p(G)$

(b) $\mathrm{Res}_H : \mathrm{S}_0^p(G) \to \mathrm{S}_0^p(H)$ c.b. & w.c. surj'n

Confirmation that algerba (and o.s.s.) agrees with $(\mathrm{A}_p(G), \cdot)$.

Functorial property: averaging over normal subgroup

$N \lhd G$ closed, $T_N u(sN) = \int_N u(sn)\, dn$, suitable u

$\mathrm{A}_p(G/N)$ acts as multipliers on $\mathrm{A}_p(G)$: $uv(s) = u(sN)v(s)$.

Theorem

(a) $\mathcal{M}_p(K)$ operator module over $\mathrm{A}_p(G/N)$

(b) $T_N : \mathcal{M}_p(K) \to \mathrm{A}_p(G/N)$ c.b., with q.o.s.s. $\rightsquigarrow$ dual o.s.s., $T_N(\mathcal{M}_p(K))$ Segal ideal in $\mathrm{A}_p(G/N)$.

(c) $T_N : \mathrm{S}_0^p(G) \to \mathrm{S}_0^p(G/N)$ c.b. & w.c.surj'n.

Confirmation that algebra (and o.s.s.) agrees with $(\mathrm{L}^1(G), *)$.

Functorial property: isomorphism

Theorem [S. '07]

G, H – l.c.gs.
$G \cong H \Leftrightarrow \exists\ \Phi : \mathrm{S}_0^p(G) \to \mathrm{S}_0^p(H)$ cts. bij'n s.t.

$$\Phi(uv) = \Phi u \cdot \Phi v \ \&\ \Phi(u * v) = \Phi u * \Phi v$$

Just one operation is not sufficent.

G – discrete, $\mathrm{S}_0^p(G) = \ell^1(G)$
$\Rightarrow (\ell^1(G), \cdot) = (\ell^1(H), \cdot) \Leftrightarrow |G| = |H|$

G – compact abelian, $\mathrm{S}_0^p(G) \cong \mathrm{S}_0^p(\hat{G}) \cong \ell^1(\hat{G})$,
$(\mathrm{S}_0^p(G), *) \cong (\ell^1(\hat{G}), \cdot)$

Primary reference

S. Öztop & N.S. p-Operator space structures on Feichtinger–Figà-Talamanca–Herz algebras, preprint, `arXiv:1208.2072`.

Thank you!