

## 1. Solutions to exercises

### 1.1. Chapter 7

2. (a) divergent.  
 (b) convergent (limit 0).  
 (c) divergent.  
 (d) convergent (limit  $2 - \frac{i}{2}$ ).  
 (e) convergent (limit 0).
3. (a) convergent.  
 (b) divergent.  
 (c) divergent.  
 (d) convergent.
13. 1.
20. (a) Pointwise convergence on  $|z| < 1$ , uniform convergence on  $|z| \leq r$  for  $r < 1$ .  
 (b) Pointwise convergence on  $|z| \leq 1$ , uniform convergence on  $|z| \leq 1$ .  
 (c) Pointwise convergence on  $\operatorname{Re}(z) \geq 0$ , uniform convergence on  $|z| \geq r$  for  $r > 0$ .
24. (a)  $\sum_{k \geq 0} (-4)^k z^k$ .  
 (b)  $\sum_{k \geq 0} \frac{1}{3 \cdot 6^k} z^k$ .  
 (c)  $\sum_{k \geq 2} \frac{k-1}{4^k} z^k$ .
25. (a)  $\sum_{k \geq 0} \frac{(-1)^k}{(2k)!} z^{2k}$ .  
 (b)  $\sum_{k \geq 0} \frac{(-1)^k}{(2k)!} z^{4k}$ .  
 (c)  $\sum_{k \geq 1} \frac{(-1)^{k+1}}{(2k-1)!} z^{2k+1}$ .  
 (d)  $\sum_{k \geq 1} \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} z^{2k}$ .
27. (a)  $\sum_{k \geq 0} (-1)^k (z-1)^k$ , convergence radius 1.  
 (b)  $\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (z-1)^k$ , convergence radius 1.
30. (a)  $\infty$  if  $|a| < 1$ , 1 if  $|a| = 1$ , and 0 if  $|a| > 1$ .  
 (b) 1.  
 (c) 1.  
 (d) 1.  
 (e)  $\infty$ .  
 (f) 1.  
 (g)  $\frac{1}{4}$ .
31. (a)  $\exp(z^2)$ .  
 (b)  $\frac{1}{(2-z)^2}$ .  
 (c)  $\frac{2z^2}{(1-z)^3}$ .

### 1.2. Chapter 8

1. (a)  $\{z \in \mathbb{C}: |z| < 1\}$ ,  $\{z \in \mathbb{C}: |z| \leq r\}$  for any  $r < 1$ .  
 (b)  $\mathbb{C}$ ,  $\{z \in \mathbb{C}: |z| \leq r\}$  for any  $r$ .  
 (c)  $\{z \in \mathbb{C}: |z-3| > 1\}$ ,  $\{z \in \mathbb{C}: r \leq |z| \leq R\}$  for any  $1 < r \leq R$ .
2. (a)  $\frac{2}{(1-z)^3}$ .  
 (b)  $\sinh(z)$ .  
 (c)  $1 + \frac{1}{z-4}$ .

3.  $\sum k \geq 0 \frac{e}{k!} (z-1)^k$ .
5.  $\sum_{k \geq 0} \frac{(-1)^k}{2^{k+1}} z^{2k+1}$ .
6. (a)  $\frac{1}{1+z^2} = \frac{1}{2} - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 + 0 \cdot (z-1)^3 + \dots$ , the convergence radius is 1.  
 (b)  $\frac{1}{e^z+1} = \frac{1}{2} - \frac{1}{4}z + 0 \cdot z^2 + \frac{1}{48}z^3 + \dots$ , the convergence radius is  $\pi$ .
10. The maximum is 3 (attained at  $z = \pm i$ ), and the minimum is 1 (attained at  $z = \pm 1$ ).
12. One Laurent series is  $\sum_{k \geq 0} (-2)^k (z-1)^{-k-2}$ , converging for  $|z-1| > 2$ .
13. One Laurent series is  $\sum_{k \geq 0} (-2)^k (z-2)^{-k-3}$ , converging for  $|z-2| > 2$ .
14. One Laurent series is  $-3(z+1)^{-1} + 1$ , converging for  $z \neq -1$ .
15.  $\frac{1}{\sin(z)} = z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + \dots$
20. (a)  $\sum_{k \geq 0} \frac{(-1)^k}{(2k)!} z^{2k-2}$ .
21.  $1 + \frac{z^2}{2} + \frac{5}{24}z^4 + \dots$
23. (a) 1.  
 (b) 3.  
 (c) 4.
24. (a)  $\pm i$ , multiplicity 4.  
 (b)  $k\pi$ ,  $k \in \mathbb{Z}$  multiplicity 2.  
 (c)  $(2k+1)i\pi$ ,  $k \in \mathbb{Z}$ , multiplicity 1.  
 (d) 0, multiplicity 3, and  $\frac{\pi}{2} + k\pi$ , multiplicity 1.
25.  $\sum_{k \geq 0} (1 + \frac{(-1)^k}{2^{k+1}})z^k$  for  $|z| < 1$ ,  $\sum_{k \geq 0} \frac{(-1)^k}{2^{k+1}} z^k - \sum_{k < 0} z^k$  for  $1 < |z| < 2$  and  $\sum_{k < 0} (1 + (-1)^k 2^{k+1})z^k$  for  $|z| > 2$ .
28. It is less than or equal to  $\frac{1}{2}$ .
29. (a)  $\sqrt{R}$ .  
 (b)  $\frac{R}{3}$ .  
 (c)  $R$ .  
 (d)  $R$ .  
 (e)  $R^2$ .

### 1.3. Chapter 9

3. (a) 1,  $i$ ,  $-i$ , of order 4, 3, 3.  
 (b)  $k\pi$ ,  $k \in \mathbb{Z} \setminus \{0\}$ , of order 1.  
 (c) 0 of order 4.  
 (d)  $ki\pi$ ,  $k \in \mathbb{Z}$ , of order 1.  
 (e)  $ki\pi$ ,  $k \in \mathbb{Z} \setminus \{0\}$ , of order 1.
7. (a) 0.  
 (b) 1.  
 (c) 4.
9. (a) One Laurent series is  $\sum_{k \geq -2} \frac{(-1)^k}{4^{k+3}} (z-2)^k$ , converging for  $0 < |z-2| < 4$ .  
 (b)  $-\frac{\pi i}{8}$ .
10. (a)  $2\pi i$  (b)  $\frac{27\pi i}{4}$ .  
 (c)  $-\frac{2\pi i}{17}$ .  
 (d)  $\frac{\pi i}{3}$ .

- (e)  $2\pi i$ .
- (f) 0.
- 11. (a)  $\sum_{k \geq 0} \frac{1}{e k!} (z + 1^k)$ .
- (b)  $\frac{2\pi i}{e^{33!}}$ .
- 13. (a)  $-\frac{1}{2}$ .
- (b) 1.
- (c) 5.
- (d)  $e$ .
- (e) 4.
- 14. (a)  $\frac{1+i}{8}$ .
- (b)  $-\frac{\pi i}{3}$ .
- (c)  $2\pi i(1 - \cos(1))$ .
- (d)  $\frac{\pi i}{3}$ .
- 15. (b) 0.
- 16. (a)  $\frac{\pi}{2}$  for  $R > 1$ , 0 for  $R < 1$ .
- (c)  $\frac{\pi}{2}$ .
- 17.  $2\pi i \frac{f(a)-f(b)}{a-b}$ .