RESIDUE CALCULUS

BO BERNDTSSON

1. The Residue Theorem

Let f be a holomorphic function in a *punctured disk* $D^0 = \{z; 0 < |z-a| < R\}$ where a is a point in the complex plane and R is a positive number. Then f has an *isolated singularity* at a. By the theorem about Laurent series expansions f can be written

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n$$

in D^0.

Definition 1. The coefficient c_{-1} in f:s Laurent series a is called *the residue* of f at a. We write $c_{-1} = Res_a(f)$.

Consider now the integral

$$I = \int_{|z-a|=r} f(z)dz$$

where $0 < r < R$. Since the Laurent series converges uniformly on the circle $|z-a| = r$ we can compute the integral term by term. Since

$$\int_{|z-a|=r} (z-a)^n dz = 0$$

if $n \neq -1$,and

$$\int_{|z-a|=r} (z-a)^{-1} dz = 2\pi i,$$

we find that $I = 2\pi i Res_a(f)$. This theorem has a far reaching generalization.

Theorem 1.1. *(***The residue theorem, Theorem 9.10 in Beck et al***). Let f be a holomorphic function in a region G, except for isolated singularities. Let γ be a positively oriented simple, closed curve that is contractible in G and avoids the singularities of f. Then*

$$\int_\gamma f dz = 2\pi i \sum_k Res_{z_k}(f)$$

where the sum is taken over all singularities inside γ.

This theorem can be used to compute a lot of integral with rather little effort, but before we give examples we need to list a few ways to compute residues.

1.1. **The computation of residues.** The simplest case is when f has a *pole of order 1* at a. This means that all the coefficients c_n with negative index, except c_{-1} are zero, so that

$$f(z) = \sum_{-1}^{\infty} c_n (z-a)^n.$$

Then clearly

$$\lim_{z \to a} (z-a) f(z) = c_{-1} + \lim_{z \to a} \sum_{0}^{\infty} c_n (z-a)^{n+1} = a_{-1} = Res_a(f).$$

Example 1. Let

$$f(z) = \frac{g(z)}{z-a},$$

where g is holomorphic near a. Then $Res_a(f) = g(a)$.

This example is however a bit too special. Even though all functions f that have a pole of order 1 at a can be written as $f = g/(z-a)$ for some holomorphic function g near a, it can be a bit problematic to find g explicitly, or at least to find the value of g at a.

Example 2. Let

$$f(z) = \frac{e^z}{\sin z}.$$

Then f has a simple pole at 0 (and also simple poles at $z = k\pi$, k integer). If we put $g(z) = e^z (z/\sin z)$, then $f = g/z$, but the formula is bit dirty, and one has to work a little bit to find $g(0)$.

In practice therefore the next proposition is useful.

Proposition 1.2. *Let f be a holomorphic function with an isolated singularity at the point a. Suppose f can be written*

$$f(z) = \frac{g(z)}{h(z)}$$

where g and h are holomorphic and $h(a) = 0$, $h'(a) \neq 0$. Then f has a simple pole at a and

$$Res_a(f) = \frac{g(a)}{h'(a)}.$$

This follows since

$$\lim_{z \to a} (z-a) f(z) = \lim_{z \to a} \frac{g(z)}{h(z)/(z-a)}.$$

Using $h(a) = 0$ we get

$$\lim_{z \to a} h(z)/(z-a) = \lim (h(z) - h(a))/(z-a) = h'(a),$$

so

$$\lim_{z \to a} (z-a) f(z) = g(a)/h'(a).$$

Going back to our example above, $f(z) = e^z/\sin z$ we find immediately that

$$Res_0(f) = e^0/\cos 0 = 1.$$

Proposition 1.2 is the most useful way to compute residues when you have a simple pole. When you have a pole of higher order, say $k > 1$, so that

$$f(z) = \sum_{-k}^{\infty} c_n(z-a)^n$$

there is unfortunately no similar trick. The only way we have is

Proposition 1.3. *Let f be holomorphic near a point a, with an isolated singularity at a, and suppose that f can be written*

$$f(z) = g(z)/(z-a)^k,$$

with g holomorphic. Then

$$Res_a(f) = g^{(k-1)}(a)/(k-1)!.$$

This is also quite easy to see. Since g is holomorphic near a it has a Taylor expansion

$$g(z) = \sum_{0}^{\infty} b_m(z-a)^m.$$

Hence the Laurent expansion of f is

$$f(z) = b_0/(z-a)^{-k} + b_1(z-a)^{1-k} + ...b_{k-1}(z-a)^{-1} + ...$$

Therefore $Res_a(f) = b_{k-1} = g^{(k-1)}(a)/(k-1)!$, which is what the proposition claims.

Example 3. Let

$$f(z) = \frac{e^z}{\sin z(z-1)^2}.$$

What is

$$\int_{|z|=2} f(z)dz?$$

To compute this we first note that there are two possible singular points, namely the points where the denominator $\sin z(z-1)^2 = 0$. These are $z = 0$ and $z = 1$. The first is a simple pole and we find the residue using Prop 1.2

$$Res_0(f) = e^0/[\cos 0(0-1)^2] = 1.$$

(Here $g(z) = e^z/(z-1)^2$ and $h(z) = \sin z$. The second singularity is a double pole and we have to use Prop 1.3.

Let $g(z) = e^z/\sin z$. Then

$$g'(1) = (e^z \sin z - e^z \cos z)/\sin^2 z)|_{z=1} = e(\sin 1 - \cos 1)/\sin^2 1 = Res_1(f).$$

Hence

$$\int_{|z|=2} f(z)dz = 2\pi i \left(1 + e(\sin 1 - \cos 1)/\sin^2 1\right).$$

Messy? Yes. Easy to compute directly? No!

2. Computation of real integrals using residues.

We will consider two types of real integrals, integrals of trigonometric functions over a period $[0, 2\pi]$, and integrals over the real axis.

2.1. **Integrals involving trigonometric functions.**

The main idea is best illustrated in an example.

Example 4. Compute for $a > 1$

$$\int_0^{2\pi} \frac{dt}{a - \cos t}.$$

The idea is two write

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z + z^{-1}}{2},$$

for $z = e^{it}$. Then

$$I = \int_0^{2\pi} \frac{dt}{a - \cos t} = \int_{|z|=1} \frac{2}{2a - (z + z^{-1})} \frac{dz}{iz}.$$

In the last step here we have considered $z = e^{it}$ as a parametrization of the unit circle. Then $dz = ie^{it}dt = izdt$, so $dt = dz/(iz)$. Next we clean up the formula and get

$$I = \int_{|z|=1} \frac{2i}{z^2 - 2az + 1} dz.$$

Where are the poles? The denominator vanishes for $z = z_0 = a + \sqrt{a^2 - 1}$ and $z = z_1 = a - \sqrt{a^2 - 1}$. Since z_0 lies outside the unit circle we only need to worry about z_1. This point lies inside the circle since by the conjugate rule

$$a - \sqrt{a^2 - 1} = \frac{a^2 - (a^2 - 1)}{a + \sqrt{a^2 - 1}} = 1/(a + \sqrt{a^2 - 1}) < 1.$$

Let $f = \frac{2i}{z^2 - 2az + 1}$. By Prop 1.2

$$Res_{z_1}(f) = \frac{2i}{2z_1 - 2a} = i/(-\sqrt{a^2 - 1}).$$

Hence

$$I = 2\pi i Res_{z_1}(f) = 2\pi/\sqrt{a^2 - 1}.$$

Here are, as exercises, a few other variations on the same theme:

1. Show that

$$\int_0^{2\pi} \frac{dt}{a + b\sin t} = 2\pi/\sqrt{a^2 - b^2}$$

if $a > b \geq 0$.

2. Show that

$$\int_0^{2\pi} \frac{\cos t dt}{a + b\cos t} = (2\pi/b)\left(1 - \frac{a}{\sqrt{a^2 - b^2}}\right).$$

(It can be reduced to example 4, but try to do it directly!)

3. Compute

$$\int_0^{2\pi} (\cos t)^{2k} dt$$

for $k = 0, 1, 2....$

4. Show that

$$\int_0^{2\pi} \frac{dt}{a + (\sin t)^2} = \frac{2\pi}{\sqrt{a(a+1)}}.$$

2.2. **Integrals over the real line.** First we look at the simplest example, which we could also compute directly.

Example 5. Compute

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

Since $(1 + x^2)^{-1}$ has the primitive function $\arctan x$ we can compute the integral directly as

$$I = \lim_{x \to \infty} \arctan x - \lim_{x \to -\infty} \arctan x = \pi/2 - (-\pi)/2 = \pi.$$

Let us now for comparison compute this integral using residues. Let

$$f(z) = \frac{1}{1 + z^2}$$

for *complex* z. We first write the integral $I = \lim_{R \to \infty} I_R$, where

$$I_R = \int_{-R}^{R} f(x)dx.$$

Then we 'close the curve' by adding a semicircle γ_R of radius R in the upper halfplane. Then we get a closed curve $\Gamma_R = [-R, R] \cup \gamma_R$. We shall see in a while that the integral of f over γ_R tends to zero, so all we need is to compute $\lim_{R \to \infty} J(R)$ where

$$J(R) = \int_{\Gamma_R} f(z)dz.$$

But $J(R)$ is an integral over a closed curve and f is holomorphic in the upper half plane, except for an isolated singularity at $z = i$. Hence the residue theorem implies that

$$J(R) = 2\pi i Res_i(f) = 2\pi i/(2i) = \pi$$

if $R > 1$. (We have also used Prop 1.2 to compute the residue at the simple pole $z = i$.) Hence

$$I = \lim I(R) = \lim J(R) = \pi$$

so we are done, except that we **need** to verify that the integral over γ_R tends to 0 as $R \to \infty$. Here is how you do that:

$$\left|\int_{\gamma_R} \frac{dz}{1+z^2}\right| \leq \max_{\gamma_R}(|\frac{1}{1+z^2}|)|\gamma_R|$$

(remember $|\gamma|$ is the length of γ). Now, to estimate the max we need to estimate the denominater $(1+z^2)$ from *below*. This we do using the reverse triangle inequality

$$|1+z^2| \geq 1 - |z^2| = 1 - R^2,$$

since $|z| = R$ on γ_R. Clearly $|\gamma_R| = \pi R$. Hence

$$\left|\int_{\gamma_R} \frac{dz}{1+z^2}\right| \leq \pi R/(1-R^2) \to 0$$

when $R \to \infty$.

What is the point of this if we already know how to compute the integrals by simpler means? The main point is that the residue method also applies in many cases when we cannot find a primitive function. The next example illustrates this, and also shows why it is so important to estimate the integral over the semicircle. If you don't, you may get the **wrong answer**!

Example 6. Compute

$$I_0 = \int_{-\infty}^{\infty} \frac{e^{ix}dx}{1+x^2}.$$

We use the same method and define Γ_R and γ_R as in example 5. The point $z = i$ is again the only pole inside Γ_R, but this time the residue is

$$Res_i\left(\frac{e^{iz}}{1+z^2}\right) = e^{-1}/(2i),$$

so the integral becomes $2\pi i e^{-1}/(2i) = \pi/e$. We only need to check that the integral over γ_R tends to 0 as R tends to ∞. This is done as before

$$\left|\int_{\gamma_R} \frac{e^{iz}dz}{1+z^2}\right| \leq \max_{\gamma_R}(\left|\frac{|e^{iz}|}{1+z^2}\right|)|\gamma_R|.$$

Now we note that

$$|e^{iz}| = e^{-y} \leq 1,$$

so we can continue the estimate exactly as before (do that!). Hence the integral really becomes π/e; our computation was correct. Next, let us compute instead

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{-ix}dx}{1+x^2}.$$

Then the residue at $z = i$ equals $e/(2i)$ since $e^{-i^2} = e$. Hence we seem to get that $I_1 = \pi e$. But this is not possible, since clearly $I_1 = \bar{I}_0 = \pi/e$. What did we do wrong? *We did not check that the integral over γ_R tends to zero.* And it doesn't.

So, how do we compute I_1? One way is of course just to note that $I_1 = \bar{I}_0$ and we already have computed I_0. That is perfectly fine, and often the easiest way, but here is how you should apply the residue method in this case:

Computation of I_1 using the residue method.

We need to change the definition of Γ_R by adding to the interval $(-R, R)$ instead a semicircle of radius R in the *lower* half plane, that we still call γ_r. Then we get that

$$\int_{-R}^{R} \frac{e^{-ix}dx}{1+x^2} + \int_{\gamma_R} \frac{e^{-iz}dx}{1+z^2} = \int_{\Gamma_R} \frac{e^{-iz}dx}{1+z^2},$$

where as before $\Gamma_R = [-R, R] \cup \gamma_R$. *Note however that in this case we must let Γ_R be oriented clockwise instead of counterclockwise.* Since the only singular point inside the curve is now $-i$ we find that

$$\int_{\Gamma_R} \frac{e^{-iz}dx}{1+z^2} = -2\pi i Res_{-i}\left(\frac{e^{-iz}dx}{1+z^2}\right).$$

(We get $-2\pi i$ instead of $2\pi i$ because the curve has the 'wrong' orientation.) Computing the residue we get

$$Res_{-i}\left(\frac{e^{-iz}dx}{1+z^2}\right) = e^{-i(-i)}/(-2i),$$

so the integral over Γ_R becomes π/e. Now we let $R \to \infty$ and check that the integral over γ_R tends to 0, leading to

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{-ix}dx}{1+x^2} = \pi/e.$$

A variation on the same theme.

We are going to compute

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

We could do it exactly as before, by adding a semicircle in the upper half plane, but then we get two singular points. Here is another way: Compute instead

$$I = \int_{0}^{\infty} \frac{dx}{1+x^4} = \lim I_R = \int_{0}^{R} \frac{dx}{1+x^4}$$

(and multiply by 2). We now construct a closed curve Γ_R by adding to $(0, R)$ a quarter circle γ_R with radius R and then a line segment l_R from iR to 0. The only singular point inside the circle is $z_0 = (1+i)/\sqrt{2}$. Then by the residue theorem

$$\int_{\Gamma_R} \frac{dz}{1+z^4} = 2\pi i Res_{z_0}(1/(1+z^4)) = 2\pi i/(4z_0^3).$$

What is this? Since $z_0^4 = -1$, $z_0^3 = -1/z_0$. Hence

$$2\pi i/(4z_0^3) = -\pi(i/2)z_0 = (\pi/2\sqrt{2})(1-i)$$

It looks like we are going to get a complex answer to a real integral, but we have still not looked at the integral over l_R. If we use the parametrization $l_R(t) = it$ with t going from R to 0, we find

$$\int_{l_R} \frac{dz}{1+z^4} = -\int_0^R \frac{idt}{1+t^4} = -iI_R!$$

After checking that the integral over γ_R tends to zero we find

$$(1-i)I = 2\pi i/(4z_0^3) = (\pi/2\sqrt{2})(1-i),$$

so

$$I == \pi/(2\sqrt{2}).$$

Here are some exercises on this:

1. Compute

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

using the method of example 5.

2. Compute

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6}.$$

3. Compute

$$\int_{-\infty}^{\infty} \frac{\cos axdx}{1+x^2}.$$

4. Prove that

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = (\pi/2)[ab(a+b)]^{-1}.$$

3. The argument principle

Recall first the argument principle from Beck et al (Theorem 9.14).

Theorem 3.1. *Let f be holomorphic in G except for poles at the points w_k. Let γ be a closed simple curve in G which is G-contractible (null-homotopic) in G and does not pass through any zero or pole of f. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = N(f,\gamma) - P(f,\gamma),$$

where N and P stand for the number of zeros and poles respectively inside γ.

We will complement this theorem by the following geometric interpretation which moreover explains the name 'argument principle'.

Look at the image of γ under the map f, Γ. If γ is parametrized by $z = \gamma(t)$, $a < t < b$, then Γ is parametrized by $w = f(\gamma(t))$. By assumption f is never zero on γ, so Γ is a closed curve which does not pass through zero. We denote by $W(f, \gamma)$ the *winding number* of Γ around zero, i e the net number of times that Γ winds around 0. This can be thought of as the amount by which the argument of f changes as you go around γ, divided by 2π. It is not so easy to give this a rigorous definition, but the meaning in concrete situations is hopefully clear.

Example 7. Let γ be the circle $|z| = 1$ and let f be z^m. Then γ can be parametrized by $z = e^{it}$, with t running from 1 to 2π. Therefore Γ is parametrized by $w = e^{imt}$, which is a circle that winds m times around the origin, so the winding number is m.

Example 8. Let γ_R be the circle $|z| = R$, with $R >> 0$, and let

$$f = z^m + a_1 z^{m-1} + ...a_m$$

be an arbitrary polynomial. If R is sufficiently large and z lies on γ_R, then $f(z) = z^m(1 + \epsilon(z))$, where

$$\epsilon(z) = a_1/z - a_2/z^2 + ...a_m/z^m$$

is small. Then the argument of f satisfies

$$arg(f(z)) = arg(z^m) + arg(1 + \epsilon(z)).$$

Since $1 + \epsilon(z)$ is all the time close to 1 as you go around γ, the net change of its argument is zero. Hence $W(f, \gamma_R) = W(z^m, \gamma_R) = m$.

Theorem 3.2. (The honest argument principle)

$$W(f, \gamma) = N(f, \gamma) - P(f, \gamma).$$

Proof. By theorem 3.1 we only need to prove that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} dz = W(f, \gamma). \tag{3.1}$$

If we parametrize γ by $z = \gamma(t)$, $0 < t < 1$ we have that

$$\int_\gamma \frac{f'}{f} dz = \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt.$$

But

$$\frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) = (d/dt) \log f(\gamma(t))$$

where we can take any branch of the logarithm. This equals in turn

$$(d/dt) \log |f(\gamma(t))| + i(d/dt) arg(f(\gamma(t))).$$

Therefore

$$\int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt = i[arg(f(\gamma(1))) - arg(f(\gamma(0)))],$$

which is i times the net amount that the argument of $f(\gamma(t))$ increases as we go around γ. Dividing by $2\pi i$, this is the winding number. $\square$

Example 9. **Yet another proof of the fundamental theorem of algebra.** Let's go back to Example 8. We have seen that the winding number of

$$f = z^m + a_1 z^{m-1} + ...a_m$$

along $\gamma_R = \{|z| = R\}$ is m, if $R >> 0$. Since f has no poles, Theorem 3.2 says that f has m zeros inside γ_R, if R is large.

Example 10. Let $\lambda > 1$. How many solutions does the equation

$$z + e^{-z} = \lambda \tag{3.2}$$

have in the right half plane?

Let γ_R be the semicircle in the right halfplane consisting of the set

$$c_R = \{\mathrm{Re}\, z > 0 \quad and \;\; |z| = R\}$$

and the interval from iR to $-iR$ on the imaginary axis. Let $h(z) = z + e^{-z} - \lambda$. First, we parametrize the interval on the imaginary axis by $z = it$ where t runs from R to $-R$. Then

$$h(it) = it + e^{-it} - \lambda = (\cos t - \lambda) + i(t - \sin t).$$

Since $\lambda > 1$, the real part of h is always negative and the imaginary part decreases from a large positive value to a large negative value. Therefore the argument variation on the interval is

$$Argvar_{[iR,-iR]}(h) \sim \pi.$$

On c_R, we parametrize by $z = z(s) = Re^{is}$. Then

$$h(z(s)) = Re^{is} + e^{-z(s)} - \lambda.$$

Since $\mathrm{Re}\, z(s) \geq 0$, $|e^{-z(s)}| = e^{-\mathrm{Re}\, z(s)} \leq 1$, so

$$h(z(s)) = Re^{is} + e^{z(s)} - \lambda = Re^{is}(1 + [+e^{z(s)} - \lambda]/(Re^{is}) = Re^{is}(1 + \epsilon(s)),$$

where $\epsilon(s)$ is very small. Hence, as in example 8,

$$Argvar_{C_R}(h) \sim \pi.$$

Therefore, the variation of the argument on all of γ_R is

$$Argvar_{\gamma_R}(h) = Argvar_{[iR,-iR]}(h) + Argvar_{C_R}(h) \sim 2\pi.$$

Since the variation of the argument along a closed curve is always an integer multiple of 2π we have in fact

$$Argvar_{\gamma_R}(h) = 2\pi,$$

so the number of solutions is 1.

B BERNDTSSON :DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY,
E-mail address: bob@chalmers.se