FOURIER AND LAPLACE TRANSFORMS

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1. Fourier series

The basic idea of Fourier analysis is to write general functions as sums (or superpositions) of trigonometric functions, sometimes called harmonic oscillations. This idea is clearest in the case of functions on a bounded interval, that for simplicity we take to be $I = (0, 2\pi)$. In that case we want to write a function $f(t)$ on I as

$$f(t) = \sum_0^\infty a_k \sin kt + b_k \cos kt.$$

Since by Eulers formulas we can express the sine and cosine functions in terms of complex exponentials e^{ikt} and e^{-ikt} we can equivalently try a to write

$$f(t) = \sum_{-\infty}^\infty c_k e^{ikt}.$$

(Note that the sum here runs over positive and negative indices.) Suppose that we have such a formula and that the sum converges uniformly. Then we can multiply by e^{-ilt} and integrate

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ilt}dt = \sum c_k \frac{1}{2\pi}\int_0^{2\pi} e^{ikt}e^{-ilt}dt = \sum c_k \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-l)t}dt.$$

Now use that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-imt}dt = 1$$

if $m = 0$ and

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-imt}dt = 0$$

if $m \neq 0$. We see that the only term that is not zero in our formula is when $k = l$ so

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt. \tag{1.1}$$

The coefficients c_k that are defined in this way are the *Fourier coefficients* of f, and we have seen that if the function f can be written as a sum of complex exponentials in a nice way, then the coefficients must be the Fourier coefficients. Conversely, a general result (from the course in Fourier analysis) says that 'nice' functions indeeed can be written in this way. Here are some examples of Fourier expansions:

$$t = \pi + i \sum_{k \neq 0} \frac{1}{k} e^{-ikt},$$

and

$$t(2\pi - t) = \frac{2\pi^2}{3} - 2 \sum_{k \neq 0} \frac{1}{k^2} e^{-ikt}.$$

Exercise: 'Prove' these formulas by computing the Fourier coefficients by the formula (1.1). □

Notice that the second formula gives with $t = 0$ that

$$\sum_{1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

The idea to represent general functions as Fourier series is very old, but still very useful. The idea from antiquity to write the orbits of the planets as a superposition of circular motions can be seen as an early example. Other natural cases are to write 'the sound of music' as a sum of tones of different frequences, or the devolopment of economy as sums of business cycles. More surprising is perhaps that Fourier's original motivation came from the study of heat transfer or that Fourier analysis lies behind medical tomography. This last application was awarded with the Nobel prize for medicine in 1979 - one of many Nobel prizes to Fourier analysis.

In this course we will only study the similar *Fourier transform* for functions on $(-\infty, \infty)$, and we only use Fourier series as a motivating introduction.

2. The Fourier transform

We shall look at functions $f(t)$ defined on the real line $(-\infty, \infty)$ and we will mostly assume that they are absolutely integrable so that

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty.$$

We shall say that such functions lie in L^1. Our functions will be continuous or even differentiable, but we also allow functions that are *piecewise* continuous or differentiable. This means that $\mathbb{R}$ can be divided into a finite collection of intervals where f is continuous or differentiable.

Example 1. The function which equals t when $-\pi < t < \pi$ and equals zero for other values of t is piecewise differentiable. The function which equals $t(2\pi - t)$ when $0 < t < 2\pi$ and is zero otherwise is piecewise differentiable and continuous everywhere (not just piecewise continuous).

For such funtions we can define their *Fourier transforms* by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-ixt} dt, \tag{2.1}$$

for x in $\mathbb{R}$. Thus, the Fourier transform of a function on $\mathbb{R}$ is again a function on $\mathbb{R}$. Compare the definition of Fourier coefficients! The main differences are that the Fourier transform is defined for functions on all of $\mathbb{R}$, and that the Fourier transform is also a function on all of $\mathbb{R}$, whereas the Fourier coefficients are defined only for integers k. Here are two fundamental theorems about the Fourier transform:

Theorem 2.1. *(***The Fourier inversion theorem***) Assume that f is in L^1 and that $\hat{f}$ is also in L^1. Then f is continuous and*

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{itx}dx \tag{2.2}$$

for all t. In particular, the function is uniquely determined by its Fourier transform.

Note the similarity with Fourier series! If one looks at the integral as a generalized sum, we see that Theorem 2.1 also expresses an almost general function as a 'sum' of complex exponentials, but instead of summing over all integers (frequencies) k we take a continuous 'sum' over *all* frequencies x in $\mathbb{R}$.

Theorem 2.2. *(***Parsevals formula***) Assume that f is in L^1 and that $\hat{f}$ is also in L^1. Then*

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx. \tag{2.3}$$

More generally, if f and g are two functions in L^1 whose Fourier transforms also lie in L^1, we have

$$\int_{-\infty}^{\infty} f\bar{g}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)}dx. \tag{2.4}$$

We look at a few examples.

Example 2. Let

$$f(t) = e^{-|t|}.$$

Then

$$\hat{f}(x) = \int_0^{\infty} e^{-(t+itx)}dt + \int_{-\infty}^0 e^{(t-itx)}dt = 1/(1+ix) + 1/(1-ix) = 2/(1+x^2).$$

Example 3. Let $g(t) = 1/(1+t^2)$. Then

$$\hat{g}(x) = \int_{-\infty}^{\infty} \frac{e^{-itx}}{1+t^2}dt.$$

This is less easy to compute directly, but fortunately we can use residue calculus! Let us first assume that $x \leq 0$. Then we can apply the residue method by adding a semicircle in the **upper** halfplane. (Compare with the notes on residues; we take the upper half plane because e^{-izx} is bounded there if $x \leq 0$!) The only singular point of the function

$$f(z) = \frac{e^{-izx}}{1+z^2}$$

in the upper half plane is $z = i$ and the residue there is

$$Res_i(f) = e^x/(2i).$$

Hence the integral becomes

$$\hat{g}(x) = 2\pi i e^x/(2i) = \pi e^x = \pi e^{-|x|}.$$

If $x > 0$ we can either do the same computation with a semicircle in the lower half plane or note that $\hat{g}(x) = \overline{\hat{g}(-x)}$. The second way is easiest, and we find that if $x \geq 0$

$$\hat{g}(x) = \overline{\hat{g}(-x)} = \overline{\pi e^{-|-x|}} = \pi e^{-|x|}.$$

So $\hat{g}(x) = \pi e^{-|x|}$ for all x in $\mathbb{R}$.

Let us now compare these two examples. The Fourier inversion theorem says that if $h = \hat{f}$ then $\hat{h}(-t) = 2\pi f(t)$. If we take $f(t) = e^{-|t|}$ we found that $h = \hat{f} = 2g$, where g is the function in the second example. By example 3, $\hat{h}(-t) = 2\hat{g}(-t) = 2\pi e^{-|t|} = 2\pi f(t)$, exactly as predicted by the inversion formula!

We next write down a few useful formulas for the Fourier transform.

Proposition 2.3. *We have:*

a. **Linearity:** *The Fourier transform is linear; if $h = af + bg$ where a and b are complex numbers, then $\hat{h} = a\hat{f} + b\hat{g}$.*

b. **Derivative:** *If f is continuous and piecewise differentiable with f' in L^1, then $\widehat{f'}(x) = ix\hat{f}(x)$.*

c. **Translates:** *If a is real and $f_a(t) = f(t-a)$, then $\hat{f}_a(x) = e^{-iax}\hat{f}(x)$.*

d. **Scaling:** *If $a \neq 0$ is real and $g(t) = f(at)$, then $\hat{g}(x) = \hat{f}(x/a)/|a|$.*

e. **Multiplication by t:** *Assume that $tf(t)$ lies in L^1. Then $\widehat{tf(t)} = i(d/dx)\hat{f}(x)$.*

Proof. We leave (a) as an exercise and prove (b) when f is (not only piecewise) differentiable. Then

$$\widehat{f'}(x) = \lim_{R\to\infty} \int_{-R}^{R} f'(t)e^{-ixt}dt =$$

$$= \lim_{R\to\infty} \left([f(t)e^{-ixt}]_{-R}^{R} + ix \int_{-R}^{R} f(t)e^{-ixt}dt \right).$$

One can prove that if f and f' lie in L^1, then $\lim f(R)$ as $R \to \infty$ equals zero (try!). This gives

$$\widehat{f'}(x) = \lim_{R\to\infty} ix \int_{-R}^{R} f(t)e^{-ixt}dt = ix\hat{f}(x).$$

The proof of (c) is easier:

$$\hat{f}_a(x) = \int_{-\infty}^{\infty} f(t-a)e^{-ixt}dt = \int_{-\infty}^{\infty} f(t)e^{-ix(t+a)}dt = e^{-iax}\hat{f}(x).$$

We also leave (d) as an exercise. Finally (e) is obtained formally by differentiating under the integral sign, but we omit the rigorous justification. □

Let us now use these rules to compute a few Fourier transforms.

Example 4. Let $f(t) = e^{-t^2/2}$. It is not so easy to compute the Fourier transform from the definition (but it can be done, using Cauchy's theorem). Instead we look at the derivative. Note that

$$f'(t) + tf(t) = 0.$$

Take the Fourier transform of this equation. Then we get by (b) and (e)

$$ix\hat{f}(x) + i(d/dx)\hat{f}(x) = 0,$$

so $\hat{f}$ solves the same equation as f! Using the method of integrating factor we find that

$$(d/dx)\left(e^{x^2/2}\hat{f}(x)\right) = e^{x^2/2}((d/dx)\hat{f} + x\hat{f}) = 0.$$

Hence

$$\hat{f}(x) = ce^{-x^2/2}.$$

What is c? One way to compute c is to note that

$$c = \hat{f}(0) = \int e^{-t^2/2}dt = \sqrt{2\pi}.$$

Or, we can apply the inversion theorem by taking the Fourier transform once more:

$$2\pi f(t) = \hat{\hat{f}}(-t) = c^2 f(-t) = c^2 f(t),$$

since f is an even function. Hence, again, $c = \sqrt{2\pi}$.

Example 5. What is the Fourier transform of

$$g(t) = (t^2 + 4t + 5)^{-1}?$$

Write $t^2+4t+5 = (t+2)^2+1$. Hence $g(t) = f(t+2)$ if $f(t) = (t^2+1)$. We know by example 3 that $\hat{f}(x) = \pi e^{-|x|}$, so by (c)

$$\hat{g}(x) = e^{2ix}\hat{f}(x) = e^{2ix}\pi e^{-|x|}.$$

Example 6. Let f be the function that equals zero if $|t| \geq 1$, equals $1-t$ when $0 < t < 1$ and equals $1+t$ when $-1 < t < 0$. What is the Fourier transform of f? This function is continuous and piecewise smooth. Its derivative $f' = g$ has Fourier transform

$$\hat{g}(x) = \int_{-1}^{0} e^{-ixt}dt - \int_{0}^{1} e^{-ixt}dt = \frac{2(\cos x - 1)}{ix}.$$

By (b)

$$\hat{f}(x) = (ix)^{-1}\hat{g}(x) = 2\frac{1-\cos x}{x^2}.$$

Example 7. Another computation of the Fourier transform of $f(t) = e^{-t^2/2}$. From the definition we get, completing squares, that

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-t^2/2}e^{-itx}dt = \int_{-\infty}^{\infty} e^{-(t+ix)^2/2}dt\ e^{-x^2/2}.$$

Our claim that $\hat{f}(x) = ce^{-x^2/2}$ where c is constant, is therefore equivalent to saying that

$$I_x = \int_{-\infty}^{\infty} e^{-(t+ix)^2/2}dt$$

does not depend on x. But, I_x can be written

$$I_x = \int_{\operatorname{Im} z = x} e^{-z^2/2}dz.$$

The fact that $I_x = I_0$ is therefore *formally* a consequence of Cauchy's integral theorem, since the curve $\operatorname{Im} z = x$ is homotopic to the curve $\operatorname{Im} z = 0$. This is of course only formal since Cauchy's theorem is about closed finite curves, but the argument can be made precise by looking at integrals over rectangles Γ_R with horizontal sides $[-R, R]$ and $[-R + ix, R + ix]$ connected by two vertical lines (draw a figure). The integral of $f(z) = e^{-z^2/2}$ over Γ_R is zero, and the proof follows if we can prove that the integrals over the vertical sides goes to zero as $R \to \infty$.

Exercise: Prove this, i e prove that

$$\lim_{R\to\infty} \int_{R}^{R+ix} f(z)dz = 0$$

for any x. □

Another formula for the Fourier transform concerns the *convolution, (faltning)* of two functions.

Definition 1. If f and g are two functions in L^1, their convolution is the function

$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds.$$

The operation of convolution is interesting in several connections. It arises e g naturally in probability theory: If f and g are probability ditsributions for two stochastic variables, then $f * g$ is the distribution of their sum. Convolutions also tend to pop up in solution formulas for differential equations, as we will see later when we discuss Laplace transforms.

Theorem 2.4. *If $h = f * g$ then*

$$\hat{h}(x) = \hat{f}\hat{g}(x).$$

Proof. By definition

$$\hat{h}(x) = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(s)g(t-s)ds)e^{-ixt}dt = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} g(t-s)e^{-ixt}dt)f(s)ds =$$

$$= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} g(t)e^{-ixt}e^{-ixs}dt)f(s)ds = \hat{g}(x)\int_{-\infty}^{\infty} f(s)e^{-ixs}ds = \hat{g}\hat{f}.$$

□

Example 8. Let $f = e^{-t^2/2}$. What is $g = f * f$?

We know that $\hat{f}(x) = \sqrt{2\pi}e^{-x^2/2}$. By the theorem $\hat{g}(x) = 2\pi e^{-x^2}$, so we want to find a function that has this Fourier transform. Since

$$e^{-x^2} = e^{-(x\sqrt{2})^2/2}$$

we find from (d) in Proposition 2.3 that

$$\sqrt{2}\sqrt{2\pi}e^{-x^2}$$

is the Fourier transform of $e^{-(t/\sqrt{2})^2/2}$. Therefore $\hat{g}$ is the Fourier transform of $\sqrt{\pi}e^{-(t/\sqrt{2})^2/2}$. Hence

$$g(t) = \sqrt{\pi}e^{-t^2/4}.$$

One of the points with the convolution operator - and indeed with all of Fourier analysis - is that it sometimes permits us to write down explicit formulas for the solutions to some differential equations. In the next example we shall study an example of how this is done.

Example 9. Consider the differential equation on the real line

$$-u''(t) + u(t) = g(t),$$

where g is a given function. We assume that g, u and u' are in L^1, so that they have Fouriertransforms; then by the equation u'' is also in L^1. By (b) in Proposition 2.3 we get that $\widehat{u'} = ix\hat{u}$ and $\widehat{u''} = -x^2\hat{u}$. Hence we get from our differential equation that

$$(1 + x^2)\hat{u} = \hat{g},$$

so

$$\hat{u} = \frac{1}{1+x^2}\hat{g}.$$

By Theorem 2.4, Example 2 and the fact that we can recover a function from its Fourier transform, this means that

$$u(t) = ((1/2)^{-1}e^{-|t|}) * g = (1/2)^{-1}\int_{-\infty}^{\infty} e^{-|s-t|}g(s)ds,$$

so we have a solution formula for the equation. Notice however that this way we only find the solutions that lie in L^1. If u solves the equation, then $u(t) + ce^t$ also solves the equation, but these solution can not be found by our formula, since the exponential function does not lie in L^1.

Exercises:

1. Compute the Fourier transforms of

a. $f(t) = 1$ if $|t| < \sigma$, $f(t) = 0$ if $|t| \geq \sigma$. (Answer: $(2\sin\sigma x)/x$.)

b. Take $\sigma = 1/2$ in the previous exercise. Use Theorem 2.4 to show that $f * f$ equals the function in Example 6.

c. The square of the function in example 6. (Answer: $(4/x^2)[1 - \sin x/x]$.)

d. $(a^2 + b^2t^2)^{-1}$ where a,b>0. (Answer: $(ab)^{-1}\pi e^{-|ax|/b}$.)

2. Prove that $f * g = g * f$.

3. Use formula (2.4) to prove that

$$\int_{-\infty}^{\infty} \frac{\sin ax \sin bx}{x^2} dx = \pi \min(a, b).$$

4. Compute

$$\int_{-\infty}^{\infty} \frac{1}{1+s^2} \frac{1}{1+(s-t)^2} ds$$

using Theorem 2.4 and Proposition 2.3 d. (Answer: $2\pi/(4+t^2)$.)

5. Compute

$$\int_{-\infty}^{\infty} \frac{ds}{(1+s^2)^2}$$

using formula (2.3). Check you answer against the previous exercise for $t = 0$.

6. Compute

$$\int_{-\infty}^{\infty} \frac{ds}{(1+s^2)^2}$$

using residues.

3. THE LAPLACE TRANSFORM

One main point of the Fourier transform is that it exchanges differentiation and multiplication by t, under suitable hypotheses

$$\widehat{f'}(x) = ix\hat{f}(x).$$

We saw an application of this when we computed the Fourier transform of $e^{-t^2/2}$. In principle one can use this to solve differential equations, as we saw in Example 9. However this does not work so well in many cases, mainly because the functions that solve our equations do not fulfil our ' suitable hypotheses'. If for example we try to solve a very simple equation

$$f' = f$$

or even

$$f' = 0$$

we get only $f = 0$ as solutions. This is because the solutions to these equations, $f = ce^t$ and $f = c$, are not in L^1, so they have no Fourier transforms. The *Laplace transform* is a way around this difficulty. It operates on functions defined only on the interval $[0, \infty)$, that do not grow faster than exponentially.

Definition 2. Let $f(t)$ be a function defined on $[0, \infty)$ which satisfies an estimate

$$|f(t)| \leq Ae^{Bt}$$

for some constants A and B. Then its Laplace transform is the function

$$\tilde{f}(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt. \tag{3.1}$$

It is defined for complex numbers s such that $\operatorname{Re} s > B$; then the integral in the definition is convergent.

First of all we note that the Laplace transform determines the function uniquely - if we know the Laplace tranform we can in principle compute the function.

Proposition 3.1. *If $\tilde{f} = 0$, then $f = 0$.*

Proof. The Laplace transform is defined for all complex s with $\operatorname{Re} s > b$. Fix some $c > b$ and look at $\tilde{f}(s)$ for $s = c + iy$, with y real. Then define a function

$$g(t) = f(t)e^{-ct}$$

for $t \geq 0$ and

$$g(t) = 0$$

for $t < 0$. Then

$$\tilde{f}(c + iy) = \int_0^\infty f(t)e^{-(c+iy)t}dt = \int_{-\infty}^\infty g(t)e^{-iyt}dt = \hat{g}(y). \tag{3.2}$$

Hence, if $\tilde{f} = 0$, then $\hat{g} = 0$, so $g = 0$ by Fourier inversion, and it follows that $f = 0$. $\square$

Looking at this argument more closely we arrive at an inversion formula for the Laplace transform.

Theorem 3.2. Inversion formula for the Laplace transform. *If* $|f(t)| \leq ae^{bt}$ *for* $t \geq 0$ *and if* $c > b$ *and*

$$\int_{-\infty}^{\infty} |\tilde{f}(c+iy)|dy < \infty,$$

then

$$f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = c} \tilde{f}(s)e^{st}ds,$$

for $t > 0$.

Proof. By the inversion formula for the Fourier transform we have that

$$g(t) = (1/2\pi)^{-1} \int_{-\infty}^{\infty} \hat{g}(y)e^{iyt}dy.$$

Now we use the formula for $\hat{g}$ in (3.2) and get

$$g(t) = (1/2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{f}(c+iy)e^{iyt}dy.$$

Then recall that $f(t) = e^{ct}g(t)$ for $t > 0$, so

$$f(t) = (1/2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{f}(c+iy)e^{(c+iy)t}dy.$$

Finally we think of $y \to s = (c+iy)$ as the parametrization of the curve $\operatorname{Re} s = c$ in the complex plane and rewrite the last integral as

$$\frac{1}{2\pi} \int_{\operatorname{Re} s = c} \tilde{f}(s)e^{st}d(s/i) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = c} \tilde{f}(s)e^{st}ds$$

(note that $ds = idy$), which completes the proof. ☐

This formula is useful, but in many cases one uses instead a table of known Laplace transforms to invert the Laplace transform. Then we will also have use for a list of basic rules of computation for the Laplace transform, similar to Proposition 2.3 (this time we do not state the exact hypotheses).

Proposition 3.3. *We have:*

a. **Linearity:** *The Laplace transform is linear; if* $h = af + bg$ *where* a *and* b *are complex numbers, then* $\tilde{h} = a\tilde{f} + b\tilde{g}$.

b. **Derivative:** *If* f *is continuous and piecewise differentiable then*

$$\widetilde{f'}(s) = s\tilde{f}(s) - f(0).$$

c. **Higher derivatives:**

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - ...f^{(n-1)}(0).$$

d. **Multiplication by exponentials:**

$$\mathcal{L}(e^{ct}f)(s) = \tilde{f}(s-c).$$

e. **Translation** *Let f be a function defined for $t \geq 0$ and let $a > 0$. Define a new function f_a, by letting $f_a(t) = 0$ if $t < a$ and $f_a(t) = f(t-a)$ if $t \geq a$. Then*

$$\mathcal{L}(f_a)(s) = e^{-sa}\tilde{f}(s).$$

f. **Multiplication by t:**

$$\mathcal{L}(tf)(s) = (-d/ds)\tilde{f}(s).$$

Let us look at some examples:

1. $\mathcal{L}(1) = 1/s$; this follows from direct computation.

2. $\mathcal{L}(t) = 1/s^2$; this follows from 1. and f. in Proposition 3.3.

3. $\mathcal{L}(e^{ct}) = 1/(s-c)$; this follows from direct computation or 1. together with d.

4.

$$\mathcal{L}(\sin ct)(s) = \frac{c}{c^2 + s^2}.$$

To see this we use that

$$\sin ct = (e^{ict} - e^{-ict})/(2i).$$

Hence by 3.

$$\mathcal{L}(\sin ct) = \left(\frac{1}{s - ic} - \frac{1}{s + ic}\right)/(2i) = \frac{c}{c^2 + s^2}.$$

5. Similarily one finds that

$$\mathcal{L}(\cos ct)(s) = \frac{s}{c^2 + s^2}.$$

Example 10. We solve the initial value problem

$$u''(t) + u(t) = 0$$

for $t > 0$, $u(0) = a$, $u'(0) = b$. We first take the Laplace transform using the rules above and get

$$s^2\tilde{u} - sa - b + \tilde{u} = 0.$$

Hence

$$\tilde{u} = a\frac{s}{s^2 + 1} + b\frac{1}{s^2 + 1}.$$

Looking in our list of transforms we see that $u(t) = a\cos t + b\sin t$. (Check that this indeed does solve the problem!)

Example 11. Solve the initial value problem

$$u''(t) + u(t) = 1$$

for $t > 0$, $u(0) = a$, $u'(0) = b$. We first take the Laplace transform using the rules above and get

$$s^2\tilde{u} - sa - b + \tilde{u} = 1/s.$$

Hence

$$\tilde{u} = a\frac{s}{s^2+1} + b\frac{1}{s^2+1} + \frac{1}{s(s^2+1)}.$$

We therefore need to find the inverse Laplace transform of

$$\frac{1}{s(s^2+1)}.$$

For this we expand our functions in partial fractions

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Altogether

$$u(t) = a\cos t + b\sin t + 1 - \cos t.$$

In order to attack more complicated problems we also need to use the convolution. We define this as before, but using the convention that *all functions are considered to be zero for* $t < 0$. This means that

$$f * g(t) = \int_0^t f(s)g(t-s)ds,$$

since

$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds = \int_0^{\infty} f(s)g(t-s)ds = \int_0^t f(s)g(t-s)ds$$

($g(t-s) = 0$ if $s > t$!).

Proposition 3.4. *Let*

$$f * g(t) = \int_0^t f(s)g(t-s)ds.$$

Then

$$\mathcal{L}(f * g) = \tilde{f}\tilde{g}.$$

Example 12. Solve the initial value problem

$$u''(t) + u(t) = g(t)$$

for $t > 0$, $u(0) = 0$, $u'(0) = 0$, where g is any given function. Arguing as in the previous examples we find that

$$\tilde{u}(s) = \frac{\tilde{g}}{s^2+1}.$$

By Proposition 3.4 we see that

$$\frac{\tilde{g}}{s^2+1} = \mathcal{L}(g * \sin t)(s).$$

Therefore

$$u(t) = \int_0^t g(s)\sin(s-t)ds.$$

Exercises:
1. Compute the Laplace transforms of the following functions:

a. $\sinh At$ (Answer: $A(s^2-A^2)^{-1}$)

b. $\cosh At$ (Answer: $s(s^2-A^2)^{-1}$)

c. t^2e^{at} (Answer: $2/(s-a)^3$)

d. $e^{at}\cos t$. (Answer: $(s-a)/((s-a)^2+1)$.)

2. (**Reality check**) Check that the inverse Laplace transform of

$$\frac{A}{A^2+s^2}$$

is really $\sin At$ using the inversion formula in Theorem 3.2 and the residue theorem.

3. Solve the following initial value problems using Laplace transforms.
a. $u'' - 2u' + 2u = 6e^{-t};\ u(0) = 0,\ u'(0) = 1$.

b. $u'' + u = g(t)$, where $g(t) = 0$ for $0 < t < \pi$ and $g(t) = 1$ for $t > \pi$, $u(0) = u'(0) = 0$.
(Answer: $u(t) = 0$ if $t < \pi$, $u(t) = 1 + \cos t$ if $t > \pi$.)

4. Find the functions that have the following Laplace transforms.
a. $s/(s^2+1)^2$ (Answer: $(t/2)(\sin t)$.)

b. $(s-a)^{-4}$ (Answer: $t^3e^{at}/3!$.)

5. Solve the following system of equations

$$x'(t) = 2x(t) - y(t)$$

$$y'(t) = 3x(t) - 2y(t)$$

for $t > 0$ with the initial conditions $x(0) = 0$ and $y(0) = 1$. (Answer: $x(t) = (e^{-t} - e^t)/2$, $y(t) = (3e^{-t} - e^t)/2$.)

4. THE Z-TRANSFORM.

The Z-transform is analogous to the Laplace transform, but operates on sequences $\{a_k\}_0^\infty$ instead of functions. It can be used to solve difference equations and to compute explicitly sequences that are recursively defined.

Definition 3. Let $\alpha = \{a_k\}_0^\infty$ be a sequence of complex numbers that grow at most exponentially, i e there are constants M and r_0 such that

$$|a_k| \leq M r_0^k$$

for $k = 0, 1, 2, ...$ Then the Z-transform of α is

$$Z(\alpha)(z) = \sum_0^\infty \frac{a_k}{z^k}.$$

Notice that if $|z| > r_0$, then $r = r_0/|z| < 1$ and $|a_k/z^k| \leq M(r_0/|z|) = Mr^k$ so the sum converges and defines a holomorphic function of z. The Z-transform is a Laurent series, but we could also have defined it as

$$\sum_0^\infty a_k w^k,$$

and instead get a holomorphic function in a disk with radius $1/r_0$. Why did we choose to define it as a power series in $1/z$ instead? I don't have the faintest idea but once this unfortunate convention has been chosen it is best to stay with it.

Example 13. Let $a_k = 2^k$. Then

$$Z(\alpha) = \sum_0^\infty (2/z)^k = \frac{1}{1 - 2/z} = \frac{z}{z-2}.$$

If $b_k = 1/k!$ we see that $Z(\beta) = e^{1/z}$.

The first rule of computation for the Z-transform is a variant of the formula for the Laplace transform of a derivative. It involves the *shift operator*.

Definition 4. If α is a sequence then $\beta = S(\alpha)$ is defined by

$$b_k = a_{k+1}, k = 0, 1, 2...$$

Then

$$Z(\beta) = \sum_0^\infty a_{k+1}/z^k = z(\sum_1^\infty a_k/z^k).$$

Therefore we get

$$Z(S(\alpha))(z) = z[Z(\alpha) - a_0]. \tag{4.1}$$

In a similar way we get that

$$Z(S^N(\alpha)) = z^N \left[Z(\alpha) - \sum_0^{N-1} a_k/z^k \right]. \tag{4.2}$$

The only other formula we need in our toolbox involves the *convolution* of two sequences.

Definition 5. Let α and β be two sequences. Then their convolution is the sequence $\gamma = \{c_k\} = \alpha * \beta$ defined by

$$c_k = \sum_0^k a_j b_{k-j}.$$

Notice the similarity of this definition with the convolution of functions on the positive halfaxis.

Proposition 4.1.

$$Z(\alpha * \beta) = Z(\alpha)Z(\beta).$$

Proof.

$$Z(\alpha)(z)Z(\beta)(z) = \sum_0^\infty a_j/z^j \sum_0^\infty b_k/z^k = \sum_0^\infty z^{-n} \sum_0^n a_j b_{n-j} = Z(\gamma)(z).$$

□

Let us now look at a simple example to see how this works.

Example 14. Let us look at the very simple difference equation

$$a_{k+1} - a_k = b_k, \quad k = 0, 1, 2...,$$

where b_k are given and we have the initial value condition $a_0 = a$. If we let α be the sequence $\{a_k\}$ and $\beta = \{b_k\}$, the equation says that

$$S(\alpha) - \alpha = \beta.$$

Taking the Z-tranform and using (4.1) we get

$$z[Z(\alpha) - a] - Z(\alpha) = Z(\beta).$$

Hence

$$Z(\alpha) = \frac{z}{z-1}a + \frac{1}{z-1}Z(\beta).$$

Now,

$$\frac{z}{z-1} = \frac{1}{1-1/z} = \sum_0^\infty z^{-k} = Z(I),$$

if I stands for the sequence that is identically equal to 1. Moreover

$$\frac{1}{z-1} = Z(I'),$$

if I' is the sequence $0, 1, 1, 1.....$ Hence

$$a_k = a + (\beta * I')_k = a + \sum_0^{k-1} b_j,$$

which of course is easy to see directly (how?).

Exercises:

1. Compute the Z-transform of the following sequences:

a. $a_k = 2^k/k!$

b. $a_k = k$

c. $a_k = k(k-1)$

d. $a_k = k^2$

2. Find the sequences that have the Z-transforms

a. $\cos 1/z$

b. $(1-z)^{-2}$

c. $(z-1)^{-1}(z-2)^{-1}$

3. Use the Z-transform to solve

$$a_{k+1} - a_k = k, \quad a_0 = 0.$$

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