13.4 The z-transform

For sequences $(x(n))_{n=0}^{\infty}$ the z-transform is defined by

$$X(z)=\sum_{n=0}^{\infty} x(n)z^{-n}$$

Inversion formula. $x(n)=\frac{1}{2\pi i}\int_{|z|=r} X(z)z^{n-1}dz=\frac{1}{n!}\left(\frac{d}{dz^{-1}}\right)^n X(z)|_{z^{-1}=0}$ (r large enough)

In practice, $x(n)$ often is determined by series expansion or by using the table below.

Properties and table of z-transforms

Below $\theta(n)=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$

	$x(n)$	$X(z)$
z1.	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$
z2.	$a\,x(n)+b\,y(n)$	$a\,X(z)+b\,Y(z)$
z3.	$x(n-k)\theta(n-k)=\begin{cases}0, & 0\leq n\leq k-1\\ x(n-k), & n\geq k\end{cases}$	$z^{-k}X(z)$
z4.	$x(n+k)$ $(k>0)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-\ldots-$ $-z\,x(k-1)$
z5.	$a^{-n}x(n)$	$X(az)$
z6.	$nx(n)$	$-zX'(z)$
z7.	$x*y(n)=\sum_{k=0}^{n} x(n-k)y(k)$	$X(z)\,Y(z)$

	$x(n)$	$X(z)$
z8.	$\delta_k(n)=\begin{cases}1, & n=k\\ 0, & n\neq k\end{cases}$	$\frac{1}{z^k}$
z9.	$\begin{cases}0, & n=0\\ a^{n-1}, & n\geq 1\end{cases}$ (a complex)	$\frac{1}{z-a}$
z10.	$\begin{cases}0, & 0\leq n\leq k-1\\ \binom{n-1}{k-1}a^{n-k}, & n\geq k\end{cases}$	$\frac{1}{(z-a)^k}$
z11.	a^n	$\frac{z}{z-a}$
z12.	$\binom{n}{k}a^{n-k}$ $[na^{n-1}]$	$\frac{z}{(z-a)^{k+1}}$ $\left[\frac{z}{(z-a)^2}\right]$
z13.	n^2a^n	$\frac{a(z+a)z}{(z-a)^3}$
z14.	$\binom{n+k}{k}a^n$	$\left(\frac{z}{z-a}\right)^{k+1}$
z15.	$a^{n-1}\sin\frac{n\pi}{2}$	$\frac{z}{z^2+a^2}$
z16.	$\frac{1}{b}r^n\sin n\theta$ $\left(a,\ b>0,\ r=\sqrt{a^2+b^2},\ \theta=\arctan\frac{b}{a}\right)$	$\frac{z}{(z-a)^2+b^2}$
z17.	$\frac{1}{b}r^n\sin n\theta$ $\left(a,\ b>0,\ r=\sqrt{a^2+b^2},\ \theta=\pi-\arctan\frac{b}{a}\right)$	$\frac{z}{(z+a)^2+b^2}$
z18.	$r^n\cos n\theta$	$\frac{z(z-r\cos\theta)}{z^2-2rz\cos\theta+r^2}$
z19.	$r^n\sin n\theta$	$\frac{rz\sin\theta}{z^2-2rz\cos\theta+r^2}$
z20.	$a^n/n!$	$e^{a/z}$

Recurrence (difference) equations

An N^{th} order *linear recurrence equation with constant coefficients* and N initial values:

(13.1) $\quad x(n+N)+a_{N-1}\,x(n+N-1)+\ldots+a_0x(n)=f(n),\ n=0,1,2,\ldots$

(13.2) $\quad x(0), x(1), \ldots, x(N-1)$ given

To find the solution, take z-transform of (13.1) and use z4 and (13.2). This gives $X(z)$, from which $x(n)$, $n=0, 1, 2, \ldots$ are uniquely determined.

TABLE 3. SOME BASIC LAPLACE TRANSFORMS

Functions are listed on the left; their Laplace transforms are on the right. a and c denote constants with $a > 0$ and $c \in \mathbf{C}$.

1.	$f(t)$	$F(z) = \mathscr{L}f(z)$
2.	$H(t-a)f(t-a)$	$e^{-az}F(z)$
3.	$e^{ct}f(t)$	$F(z-c)$
4.	$f(at)$	$a^{-1}F(a^{-1}z)$
5.	$f'(t)$	$zF(z) - f(0)$
6.	$f^{(k)}(t)$	$z^k F(z) - \sum_0^{k-1} z^{k-1-j} f^{(j)}(0)$
7.	$\int_0^t f(s)\,ds$	$z^{-1}F(z)$
8.	$tf(t)$	$-F'(z)$
9.	$t^{-1}f(t)$	$\int_z^\infty F(w)\,dw$
10.	$f * g$	FG
11.	$t^\nu e^{ct}\quad (\mathrm{Re}\,\nu > -1)$	$\Gamma(\nu+1)/(z-c)^{\nu+1}$
12.	$t^n e^{ct}\quad (n = 0, 1, 2, \ldots)$	$n!/(z-c)^{n+1}$
13.	$(t+a)^{-1}$	$e^{az}E_1(az)$
14.	$\sin ct$	$c/(z^2+c^2)$
15.	$\cos ct$	$z/(z^2+c^2)$
16.	$\sinh ct$	$c/(z^2-c^2)$
17.	$\cosh ct$	$z/(z^2-c^2)$
18.	$\sin\sqrt{at}$	$\sqrt{\pi a/4z^3}\, e^{-a/4z}$
19.	$t^{-1}\sin\sqrt{at}$	$\pi\,\mathrm{erf}\,(\sqrt{a/4z}\,)$
20.	$e^{-a^2t^2}$	$(\sqrt{\pi}\,/2a)e^{z^2/4a^2}\,\mathrm{erfc}(z/2a)$
21.	$\mathrm{erf}\,at$	$z^{-1}e^{z^2/4a^2}\,\mathrm{erfc}(z/2a)$
22.	$\mathrm{erf}\sqrt{t}$	$1/z\sqrt{z+1}$
23.	$e^t\,\mathrm{erf}\sqrt{t}$	$1/(z-1)\sqrt{z}$
24.	$\mathrm{erfc}(a/2\sqrt{t}\,)$	$z^{-1}e^{-a\sqrt{z}}$
25.	$t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}\,e^{a/4z}\,\mathrm{erfc}(\sqrt{a/4z}\,)$
26.	$t^{-1/2}e^{-a^2/4t}$	$\sqrt{\pi/z}\,e^{-a\sqrt{z}}$
27.	$t^{-3/2}e^{-a^2/4t}$	$2a^{-1}\sqrt{\pi}\,e^{-a\sqrt{z}}$
28.	$t^\nu J_\nu(t)\quad (\nu > -\frac{1}{2})$	$2^\nu\pi^{-1/2}\Gamma(\nu+\frac{1}{2})(z^2+1)^{-\nu-(1/2)}$
29.	$J_0(\sqrt{t}\,)$	$z^{-1}e^{-1/4z}$