Fourieranalys F2 / Kf2
MVE030, TMA132

Tentamen 060831, Lösningar.

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1. Randvillkoret är ohomogent, därför gör vi ett förberedelsesteg:

$v(r,t) = 1$, $u(r,t) = v(r,t) + w(r,t)$.

Funktionen $w(r,t)$ satisfierar ekvationen

$$w_t = 4\Delta w, \quad \text{med randv. } w(1,t) = 0,$$

$$w(r,0) = r^2 - 1.$$

Separation av variabler:

$$w(r,t) = R(r)\cdot T(t),$$

$$\frac{R'' + rR'}{R} = \frac{T'}{4T} = -\mu^2.$$

$$R'' + rR' + \mu^2 R = 0, \quad R(0) \text{ ändlig}, \; R(1) = 0$$

Besselekvation.

Lösningar:

$R_n(r) = J_0(\lambda_n r)$, λ_n - nollställen av J_0.

$\mu_n = \lambda_n$

$$T_n(t) = T_n(0)\, e^{-\lambda_n t}$$

För att hitta $T_n(0)$, använder vi begynnelsevillkoret: $T_n(0) = \dfrac{\int_0^1 (r^2-1) J_0(\lambda_n r)\, r\, dr}{\int_0^1 J_0(\lambda_n r)^2\, r\, dr}$

Nämnaren är lika med $\frac{1}{2} J_1(\lambda_n)^2$, täljaren beräknas som

$$\lambda_n^{-2}\int_0^{\lambda_n} (s^2\lambda_n^{-2} - 1) J_0(s)\, s\, ds = \left(J_0(s)\, s = (J_1(s)s)' \right)$$

$$= \lambda_n^{-2}\int_0^{\lambda_n} (s^2\lambda_n^{-2} - 1)(J_1(s)s)'\, ds \overset{\text{part. int}}{=}$$

$$= -\lambda_n^{-2} \int_0^{\lambda_n} 2 s \lambda_n^{-2} J_1(s)\, s\, ds$$

$$= \left(J_1(s) s^2 = (J_2(s) s^2)' \right)$$

$$- 2 \int_0^{\lambda_n} (J_2(s) s^2)' \, ds = -2 J_2(\lambda_n) \cdot \lambda_n^2 .$$

2a. Vi beräknar andraderivatan:

$$((x^2-1)^{n+1})' = 2(n+1)x(x^2-1)^n$$

$$((x^2-1)^{n+1})'' = 2(n+1)\left[(x^2-1)^n + 2nx^2(x^2-1)^{n-1}\right]$$

Sätter $x^2 = x^2 - 1 + 1$ och kommer till

$$((x^2-1)^{n+1})'' = 2(n+1)(2n+1)(x^2-1)^n + 4(n+1)n(x^2-1)^{n-1}$$

Sätter den derivatan i definitionen av Legendrepolynomen:

$$P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!} \frac{d^{n-1}}{dx^{n-1}} \frac{d^2}{dx^2} (x^2-1)^{n+1}$$

$$= \frac{(2n+1)}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n + P_{n-1}(x) ;$$

deriverar en gång till och kommer till

$$P_{n+1}' - P_{n-1}' = (2n+1) P_n . \qquad (*)$$

Utvecklingen: $f(x) = x, \; x > 0, \; 0, \; x < 0.$

$$f(x) = \sum c_n P_n(x) ;$$

$$c_n = \frac{\int_0^1 P_n(x) f(x)}{\int_0^1 P_n(x)^2 dx} = (2n+1) \int_0^1 x P_n(x)\, dx .$$

③

För att beräkna integralen använder vi formeln (*), för $n \geq 1$

$$\int_0^1 x\, P_n(x)\,dx = \int_0^1 x\left(P'_{n+1} - P'_{n-1}\right)dx$$

partiell integr.

$$= -\int_0^1 \left(P_{n+1}(x) - P_{n-1}(x)\right)dx.$$

För att hitta $\int_0^1 P_k(x)\,dx$ använder (*) en gång till, $k \geq 1$

$$\int_0^1 P_k(x)\,dx = \int_0^1 \left(P'_{k+1} - P'_{k-1}\right)dx$$

$$= P_{k+1}(1) - P_{k-1}(1) - P_{k+1}(0) + P_{k-1}(0).$$

Eftersom $P_n(1) = 1$, ~~$P_n(0) = (-1)^n$~~

och $P_{2l+1}(0) = 0$, $P_{2l}(0) = \dfrac{(-1)^l (2l)!}{2^{2l}(l!)^2}$

hittar vi att

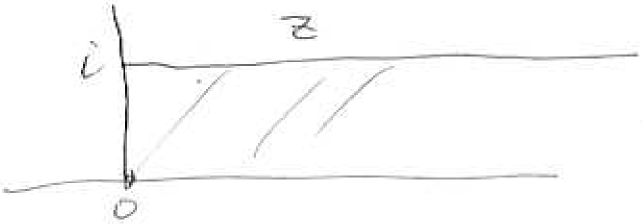
$$\int_0^1 P_k(x)\,dx = 0, \text{ för jämna } k,$$

$$\int_0^1 P_{2l+1}(x)\,dx = \frac{(-1)^{l+1}(2(l+1))!}{2^{2l+2}((l+1)!)^2} - \frac{(-1)^l (2l)!}{2^{2l}(l!)^2};$$

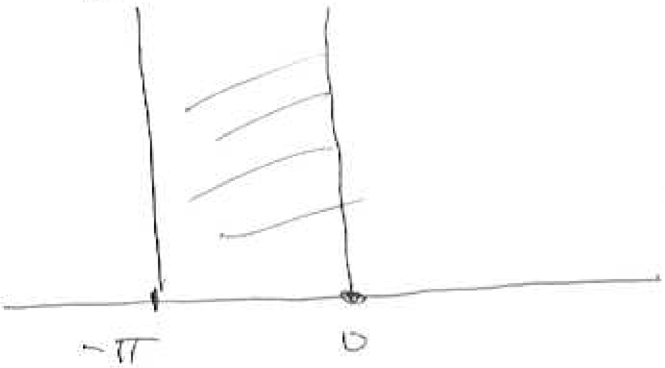
$c_n = (2n+1)\left(P_{n-2}(0) - 2P_n(0) + P_{n+2}(0)\right)$

För $n=0$, $n=1$ – separata beräkningar, med konkreta värden av $P_0(x) = 1$, $P_1(x) = x$.

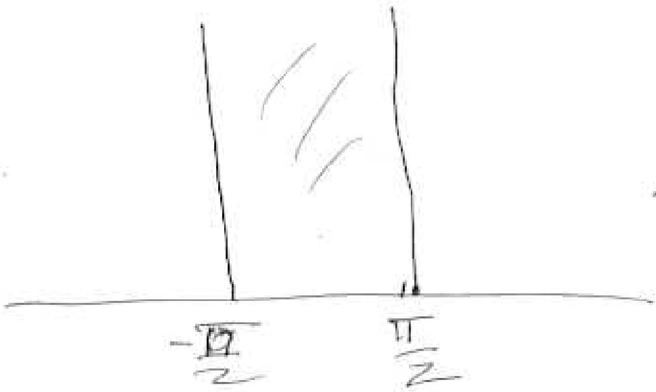
2b. Halvbandet: $x > 0$, $y \in (0,1)$ transformeras till övre halvplanet:



$\xrightarrow{w_1 = \pi z \cdot i}$



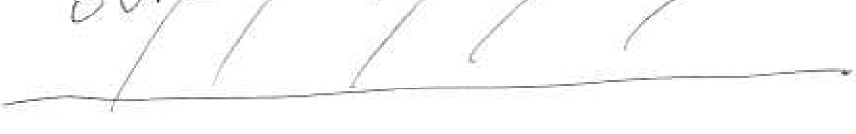
$\xrightarrow{w_2 = w_1 + \frac{\pi}{2}}$



$\longrightarrow$ ~~sh w_1~~

$w = -\sin w_1$

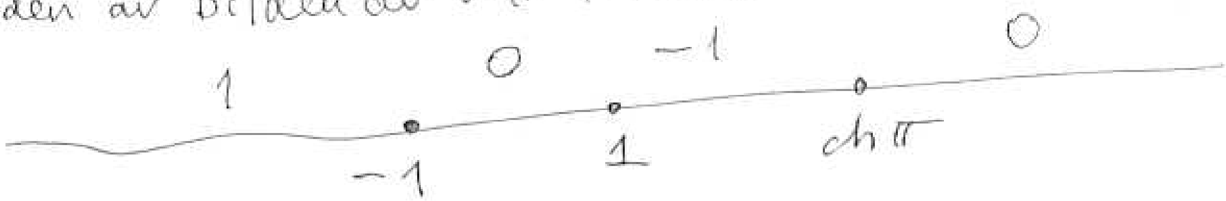
övre halvplanet.



$$w = f(z) = -\sin w_1 = -\sin(w_1 + \tfrac{\pi}{2})$$
$$= -\sin\left(\pi z \cdot i + \tfrac{\pi}{2}\right) =$$
$$\cos(\pi z \cdot i).$$

$f(0) = 1$, $f(i) = -1$, $f(1) = \operatorname{ch} \pi$

värden av bilden av vår funktion $u(z)$:



$$\cancel{V}(w) = A\arg(\cancel{w}+1) + B\arg(\cancel{w}-1) + \cancel{C\arg(w-\cosh\pi)} + C\arg(w-\cosh\pi) + D\pi$$

A, B, C, D hittar:

$$\left.\begin{aligned} \pi(A+B+C+D) &= 1 \\ \pi(B+C+D) &= 0 \\ \pi(C+D) &= -1 \\ \pi D &= 0 \end{aligned}\right\} \qquad \begin{aligned} D &= 0 \\ C &= -1/\pi \\ B &= -1/\pi \\ A &= 1/\pi \end{aligned}$$

$$V(w) = \frac{1}{\pi}\left(\arg(w+1) - \arg(w-1) + \arg(w + \cancel{\cosh}\,\pi)\right)$$

$$u(z) = V(w(z)) = V(\overset{\cos}{\cancel{\cosh}}(\pi z)).$$

3. $u_{xx} + u_{yy} - 2u = 0 \quad , x > 0 , \; y \in (0,1)$

$$u_x(0,y) = u_y(x,0) = 0, \quad u(x,1) = \begin{cases} 1, & x < c \\ 0, & x > c \end{cases}$$

Randvillkoret i $x = 0$ är $u_x(0,y) = 0$.

Vi gör Cos - Fourier transformation i x-led, därför att derivatan av Cos i $x = 0$ är 0.

$$U(\xi, y) = \int_0^\infty u(x,y) \cos \xi x \, dx.$$

$$u(x,y) = \frac{2}{\pi} \int_0^\infty U(\xi, y) \cos \xi x \, d\xi$$

$$u_x(x,y) = -\frac{2}{\pi} \int_0^\infty \xi \, U(\xi, y) \sin \xi x \, d\xi$$

$$u_{xx}(x,y) = -\frac{2}{\pi} \int_0^\infty \xi^2 \, U(\xi, y) \cos \xi x \, d\xi$$

$$u_{yy}(x,y) = \frac{2}{\pi} \int_0^\infty U_{yy}(\xi, y) \cos \xi x \, d\xi$$

Sätter in i ekvationen:

$$\frac{2}{\pi} \int_0^\infty \left(U_{yy} - \xi^2 U - 2U \right) \cos \xi x \, d\xi = 0$$

$$U_{yy} - \xi^2 U - 2U = 0$$

Randvillkoren för U:

för $y = 0$: $U_y(0) = 0$

för $y = 1$:

$$U(\xi, 1) = \int_0^\infty u(x,1) \cos x\xi \, dx = \int_0^c \cos x\xi \, dx$$

$$= \frac{\sin c\xi}{\xi}.$$

$$V_{yy} - (\xi^2+2)V = 0$$

$$V_y(\xi,0) = 0, \quad V(\xi,1) = \frac{\sin c\xi}{\xi}$$

$$V = A e^{\sqrt{\xi^2+2}\,y} + B e^{-\sqrt{\xi^2+2}\,y}$$

$$V_y(\xi,0): \quad A - B = 0$$

$$V = A\cosh\left(\sqrt{\xi^2+2}\,y\right)$$

$$V(\xi,1) = A\cosh\left(\sqrt{\xi^2+2}\cdot 1\right) = \frac{\sin c\xi}{\xi}$$

$$A = \frac{\sin c\xi}{\xi}\cdot\frac{1}{\cosh\sqrt{\xi^2+2}}$$

$$V(\xi,y) = \frac{\sin c\xi}{\xi}\cdot\frac{\cosh\left(\sqrt{\xi^2+2}\,y\right)}{\cosh\left(\sqrt{\xi^2+2}\right)}.$$

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin c\xi}{\xi}\,\frac{\cosh\left(\sqrt{\xi^2+2}\,y\right)}{\cosh\left(\sqrt{\xi^2+2}\right)}\cos x\xi\,d\xi.$$

4. $u_{tt} = u_{xx} + u \quad , 0 < x < \pi , \; t > 0$

$u(0,t) = 0, \quad u_x(\pi, t) = 1$

$u(x,0) = \frac{x^2}{2} \quad , \; u_t(x,0) = 0.$

Randvillkoret är ohomogent. Förberedelse-steget:

$v(x,t) = \frac{x^2}{2} \quad , \; u(x,t) = v(x,t) + w(x,t).$

$w_{tt} = w_{xx} + \frac{x^2}{2} + 1 + w$

$w(0,t) = 0, \quad w_x(\pi, t) = 0$

$w(x,0) = 0, \quad w_t(x,0) = 0.$

Separerar variabler:

$w(x,t) = X(x)T(t).$

$$\frac{T''}{T} = \frac{X''}{X} + 1 = -\lambda^2 + 1$$

$$X''(x) + \lambda^2 X(x) = 0 \; ; \; X(0) = 0, \; X'(\pi) = 0.$$

egenfunktioner:

$$X_n(x) = A \sin\left(\frac{(2n+1)}{2} x\right), \quad n = 0, 1, 2 \ldots$$

Normering $A = \sqrt{\frac{2}{\pi}}$, $\lambda_n = \frac{2n+1}{2}$

Lösningen söks i formen

$$w(x,t) = \sum_{n=0}^{\infty} T_n(t)\, X_n(t)$$

Sätter in i ekvationen

$$\sum T_n''(t) X_n(x) = \sum T_n(t) X_n''(x) + \sum T_n(t)\, X_n(x) + \frac{x^2}{2} + 1 .$$

Kommer ihåg att $X_n'' = -\lambda_n^2 X_n$, multiplicerar med X_k och integrerar

$$T_n''(t) + T_n(t)(\lambda_n^2 - 1) = \int_0^{\pi} \left(\frac{x^2}{2} + 1\right) X_n(x)\, dx = C_n .$$

Begynnelsevillkoren:

$T_n(0) = 0$, $T_n'(0) = 0$

Löser ordinära differentialekvationen,

$$T_n(t) = \frac{1}{\lambda_n^2 - 1}\left(\cos\left(\sqrt{\lambda_n^2 - 1}\right) t) - 1 \right).$$

5. Vi har $f(\theta) = \theta^2$, $-\pi < \theta < \pi$

$$f(\theta) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta$$

Integrerar. $c_0 = 0$, $C_0 = \int_{-\pi}^{\pi} F(\theta)d\theta = 0$.

$$F(\theta) = \int_0^{\theta} f(\varphi)\,d\varphi = \frac{\theta^3}{3}$$

$$F(\theta) = \frac{\pi^2}{3}\theta + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\theta.$$

$$\frac{\theta^3}{3} - \frac{\pi^2}{3}\theta = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\theta$$

Integrerar igen:

$$G(\theta) = \int_0^{\theta} \left(\frac{\varphi^3}{3} - \frac{\pi^2}{3}\varphi\right) d\varphi = \frac{\theta^4}{12} - \frac{\pi^2\theta^2}{6}$$

$$\int_{-\pi}^{\pi} G(\theta)d\theta = 2\frac{\pi^5}{60} - 2\pi^2\frac{\pi^3}{18} = -\frac{7}{90}\pi^5$$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\theta)d\theta = -\frac{\pi^4 \cdot 7}{180}$$

$$\frac{\theta^4}{12} - \frac{\pi^2\theta^2}{6} = -\frac{\pi^4 \cdot 7}{180} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \cos n\theta.$$

Det är vår formel. Det är samma som för $\theta = 0$ för $\theta = 4\pi$ summan är samma som för $\theta = 0$, eftersom summan är 2π-periodisk.

$$\frac{7\pi^4}{15 \cdot 48}$$

6. $u_{xx} + u_{yy} = 0$, $u(x,0) = \sin x - 2\sin 2x + 3\sin(3x)$

$u(x, 2\pi) = 0$, $u(0, y) = 0$, $u(\pi, y) = 0$.

F-serie i x-led.

vi väljer sin-serie.

$$u(x,y) = \sum_{n=1}^{\infty} f_n(y) \sin nx$$

Sätter in i ekvationen.

$$u_{xx} = -\sum_{n=1}^{\infty} f_n(y)\, n^2 \sin(nx)$$

$$u_{yy} = \sum_{n=1}^{\infty} f_n''(y) \sin(nx)$$

$$\sum \left(f_n''(y) - f_n(y) n^2\right) \sin(nx) = 0$$

$$f_n'' - n^2 f_n = 0$$

för $y = 2\pi$: $\sum f_n(2\pi) \sin nx = 0$,

$f_n(2\pi) = 0$;

för $y = 0$: $\sum_1 f_n(0) \sin nx = \sin x - 2\sin 2x + 3\sin 3x$

$f_1(0) = 1$, $f_2(0) = -2$, $f_3(0) = 3$,

alla andra $f_n(0) = 0$.

Löser ~~system~~ ekvationen.

$f_1'' - f_1 = 0$, $f_1(0) = 1$, $f_1(2\pi) = 0$

$f_2'' - 4f_2 = 0$, $f_2(0) = -2$, $f_2(2\pi) = 0$

$f_3'' - 9f_3 = 0$, $f_3(0) = 3$, $f_3(2\pi) = 0$

$f_n'' - n^2 f_n = 0$, $f_n(0) = 0$, $f_n(2\pi) = 0$, $n > 3$.

$$f_1(y) = -\frac{\sinh(2\pi - y)}{\sinh 2\pi}$$

$$f_2(y) = \frac{2\sinh\left(2(2\pi - y)\right)}{\sinh(4\pi)}$$

$$f_3(y) = -\frac{3\sinh(3(2\pi - y))}{\sinh(6\pi)}$$

$$f_n(y) = 0, \quad n > 3$$