ΕΘ 12. $f(t) = \int_0^1 \sqrt{\omega}\, e^{\omega^2} \cos\omega t \, d\omega =$

$$\left(= \frac{1}{2} \int_0^1 \sqrt{\omega}\, e^{\omega^2} e^{i\omega t} d\omega + \frac{1}{2} \int_0^1 \sqrt{\omega}\, e^{\omega^2} e^{-i\omega t} d\omega \right)$$

$$\int_{-\infty}^{\infty} |f'(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |i\omega \hat{f}(\omega)|^2 d\omega =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega$$

$$f(t) = \frac{1}{2} \int_{-1}^{1} \sqrt{|\omega|}\, e^{\omega^2} e^{i\omega t} d\omega =$$

$$= \pi \mathcal{F}^{-1}[g(\omega)](t) \qquad g(\omega) = \begin{cases} \sqrt{|\omega|}\, e^{\omega^2} & |x|<1 \\ 0 & |x|>1 \end{cases}$$

$$\hat{f}(\omega) = \pi g(\omega)$$

$$\Rightarrow \int_{-\infty}^{\infty} |f'(t)|^2 dt = \frac{\pi}{2} \int_{-\infty}^{\infty} \chi_{[-1,1]}(\omega) |\omega| \omega^2 e^{2\omega^2} d\omega$$

$$= \frac{\pi}{\cancel{2}} \cancel{2} \int_0^1 \omega^3 e^{2\omega^2} d\omega = \frac{\pi}{2} \int_0^1 \xi e^{2\xi} d\xi =$$

$$= \frac{\pi}{4} \left[\xi e^{2\xi} \right]_0^1 - \frac{\pi}{4 \cdot 2} \left[e^{2\xi} \right]_0^1 =$$

$$= \frac{\pi}{4} e^2 - \frac{\pi}{8} e^2 + \frac{\pi}{8} = \frac{\pi}{8} (e^2 + 1)$$

(EÖ16) $\frac{1}{1+t^2} \longrightarrow \frac{t}{(4+t^2)^2}$ (se EÖ6a)

$$\hat{h}(\xi)\,\pi e^{-|\xi|} = -\frac{\pi}{\underset{2a}{4}}\, i\xi e^{-2|\xi|}$$

$$\Rightarrow \hat{h}(\xi) = -\frac{1}{4} i\xi e^{-|\xi|} =$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{2} i\xi e^{-|\xi|}\right]$$

$$\Rightarrow h(t) = \frac{t}{(1+t^2)^2}$$ ej kausalt

$$\cos\omega t \longrightarrow \mathrm{Re}\left(\hat{h}(\omega)e^{i\omega t}\right) =$$

h reell $= \mathrm{Re}\left(-\frac{1}{4} i\omega e^{-|\omega|} e^{i\omega t}\right) =$

$$= \frac{1}{4}\omega e^{-|\omega|} \sin\omega t$$

? stabilt : $\int_{-\infty}^{\infty} \frac{t}{(1+t^2)^2}\,dt < \infty$ ja

(poler : endast för kausala s.)

(EÖ20) $\hat{x}_\nu = \sum_{n=0}^{N-1} e^{-2\pi i\nu n/N} x_n \quad (0 \le \nu < N)$

$$= \sum_{n=0}^{N-1} \sin\frac{n\pi}{N} e^{-2\pi i\nu n/N} = \sum_{n=0}^{N-1} \frac{e^{\frac{i\pi n}{N}} - e^{-\frac{i\pi n}{N}}}{2i} e^{-\frac{2\pi i\nu n}{N}}$$

$$= \frac{1}{2i}\sum_{n=0}^{N-1}\left(e^{-\frac{\pi i n}{N}(2\nu-1)} - e^{-\frac{\pi i n}{N}(2\nu+1)}\right) =$$

$$= \frac{1}{2i}\left[\frac{e^{-\pi i(2\nu-1)} - 1}{e^{-\frac{\pi i(2\nu-1)}{N}} - 1} - \frac{e^{-\pi i(2\nu+1)} - 1}{e^{-\frac{\pi i(2\nu+1)}{N}} - 1}\right] =$$

$$= -2\frac{1}{2i} \frac{e^{-\frac{\pi i 2\nu}{N}}\left(e^{-\frac{\pi i}{N}} - e^{\frac{\pi i}{N}}\right)}{e^{-\pi i 4\nu/N} - e^{-\frac{\pi i 2\nu}{N}}\left(e^{\frac{\pi i}{N}} - e^{-\frac{\pi i}{N}}\right) - 1} = \frac{\sin\frac{\pi}{N}}{\cos\frac{2\nu\pi}{N} - \cos\frac{\pi}{N}}$$

23. Bestäm samtliga egenvärden och tillhörande egenfunktioner till Sturm-Liouville-problemet.

$$\begin{cases} f'' + \lambda f = 0 & 0<x<a \\ f(0) - f'(0) = 0, & f(a) + 2f'(a) = 0. \end{cases}$$

Lösning:

$\lambda = 0$: $f''=0$, $f(x) = c_1 x + c_2$; $f(0) - f'(0) = c_2 - c_1 = 0$, $c_2 = c_1$.
$f(a) + 2f'(a) = c_1 a + c_2 + 2c_2 = (a+3)c_1 = 0$, $c_1 = c_2 = 0$; $\lambda = 0$ ej egenvärde.

$\lambda \neq 0$: Sätt $\lambda = \nu^2$, där $\nu = \sqrt{\lambda} > 0$ om $\lambda > 0$ och $\nu = i\sqrt{-\lambda} = i\mu$, $\mu > 0$, om $\lambda < 0$. $f(x) = c_1 \cos\nu x + c_2 \sin\nu x$; $f'(x) = \nu(c_2 \cos\nu x - c_1 \sin\nu x)$

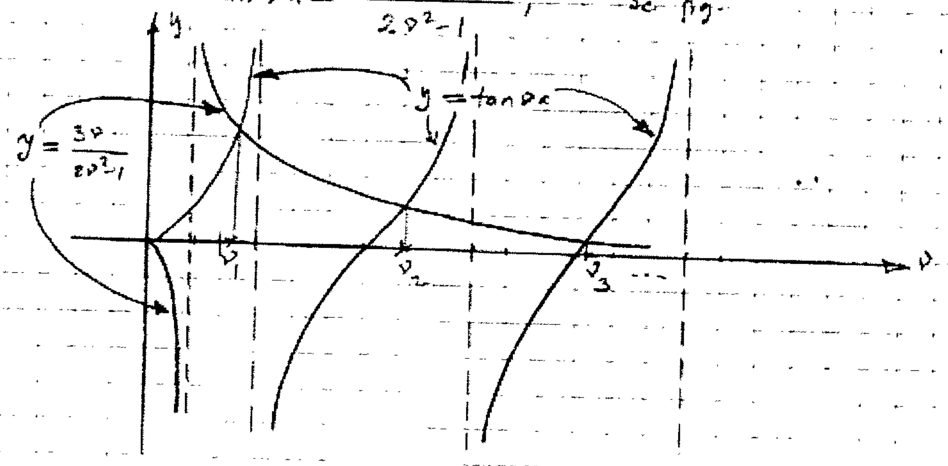
$f(0) - f'(0) = c_1 - \nu c_2 = 0$, $c_1 = \nu c_2$

$f(a) + 2f'(a) = c_1 \cos\nu a + c_2 \sin\nu a + 2\nu(c_2 \cos\nu a - c_1 \sin\nu a)$

$= 3\nu c_2 \cos(\nu a) + (1 - 2\nu^2) c_2 \sin(\nu a) = 0$, $c_2 = 0 \Rightarrow$

$3\nu \cos\nu a = (2\nu^2 - 1)\sin\nu a$, $\tan\nu a = \dfrac{3\nu}{2\nu^2 - 1}$.

För $\nu > 0$, låt $\nu_1, \nu_2, \nu_3, \ldots$, vara de positiva rötterna till ekvationen $\tan\nu a = \dfrac{3\nu}{2\nu^2-1}$, se fig.



Om $\nu = i\mu$ får ekv. $i\tanh(\mu a) = \dfrac{3i\mu}{-2\mu^2-1}$, $\tanh\mu a = -\dfrac{3\mu}{2\mu^2+1}$

som saknar rötter $\mu > 0$ (V.L. > 0, H.L. < 0).

Egenvärdena är alltså ν_k^2, där ν_k är de positiva rötterna till $\tan\nu a = \dfrac{3\nu}{2\nu^2-1}$, och egenfunktionerna är $f_k(x) = \nu_k \cos\nu_k x + \sin\nu_k x$.

29. Lös problemet
$$\begin{cases} \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, & 0<x<1, \quad t>0 \\ u(0,t) = t+1, & u(1,t)=0, \quad t>0 \\ u(x,0) = 1-x, & 0<x<1. \end{cases}$$

Lösning: Tag en funktion $\tilde{u}(x,t)$ som uppfyller randvillkoren t.ex. $\tilde{u}(x,t) = (t+1)(1-x)$. Sätt $v = u-\tilde{u}$; då satisfierar v

$$\begin{cases} \frac{\partial v}{\partial t} - 2\frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial \tilde{u}}{\partial t} - 2\frac{\partial^2 \tilde{u}}{\partial x^2}\right) = 0-(1-x)-2\cdot 0 = x-1. \\ v(0,t) = 0, \quad v(1,t)=0 \\ v(x,0) = 1-x-(1-x) = 0. \end{cases}$$

Variabelseparation i motsvarande homogena ekv. ger egenvärdesproblemet:
$X''(x) = -\lambda X(x)$, $X(0)=X(1)=0$, med lösningar $\lambda_n = (n\pi)^2$, $X_n(x) = \sin(n\pi x)$, $n\geq 1$.

Utveckla högerledet i Fourierserie m.a.p. dessa egenfunktioner,
$x-1 = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$, och ansätt $v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin(n\pi x)$.

Vi får $c_n = \frac{1}{\|X_n\|^2}\int_0^1 (x-1)X_n(x)\,dx = \frac{1}{1/2}\int_0^1 (x-1)\sin n\pi x\,dx = 2\left[(x-1)\frac{-\cos n\pi x}{n\pi}\right]_0^1$
$+2\int_0^1 \frac{\cos n\pi x}{n\pi}\,dx = -\frac{2}{n\pi} + 2\left[\frac{\sin n\pi x}{(n\pi)^2}\right]_0^1 = -\frac{2}{n\pi}.$

$$\begin{cases} \frac{\partial v}{\partial t} - 2\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty}\left[v_n'(t) + 2(n\pi)^2 v_n(t)\right]\sin n\pi x = x-1 = \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin n\pi x \\ v(x,0) = \sum_{n=1}^{\infty} v_n(0)\sin n\pi x = 0. \end{cases}$$

$$\Rightarrow \begin{cases} v_n'(t) + 2n^2\pi^2 v_n(t) \overset{(*)}{=} -\frac{2}{n\pi}, \\ v_n(0) = 0 \end{cases}$$

$I.F. = e^{2n^2\pi^2 t}$ $\xrightarrow{\cdot(*)}$

$\frac{d}{dt}\left(e^{2n^2\pi^2 t} v_n(t)\right) = -\frac{2}{n\pi}e^{2n^2\pi^2 t}$

Byt t: t mot s & $\int_0^t$ får

$e^{2n^2\pi^2 t} v_n(t) - v_n(0) = -\frac{1}{n^3\pi^3}e^{2n^2\pi^2 t} + \frac{1}{n^3\pi^3}$

$$\Rightarrow v_n(t) = \frac{1}{n^3\pi^3}\left(e^{-2n^2\pi^2 t} - 1\right)$$

$$v(x,t) = \sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}\left(e^{-2n^2\pi^2 t}-1\right)\sin n\pi x$$

$$u(x,t) = (t+1)(1-x) + \sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}\left(e^{-2n^2\pi^2 t}-1\right)\sin n\pi x$$

36. Bestäm det polynom $P(x)$ av högst andra graden som gör integralen
$$\int_0^\infty [\sqrt{x} - P(x)]^2 e^{-x}\,dx\,,$$
så liten som möjligt.

Lösning: Använd Laguerrepolynom som är ortogonala på $(0,\infty)$ med viktfunktion e^{-x} ($w(x) = x^\alpha e^{-x}$ med $\alpha = 0$). Skriv
$$P(x) = c_0 L_0(x) + c_1 L_1(x) + c_2 L_2(x),$$
Då blir $\int_0^\infty [\sqrt{x} - \sum_{k=0}^{2} c_k L_k(x)]^2 e^{-x}\,dx = \|\sqrt{x} - \sum_{k=0}^{2} c_k L_k(x)\|^2_{w(x)}$
så liten som möjligt, då
$$c_k = \frac{1}{\rho_k}\int_0^\infty \sqrt{x}\, L_k(x) e^{-x}\,dx = \int_0^\infty \sqrt{x}\, L_k(x)\,dx, \quad \text{ty}$$
$$\rho_k = \frac{1}{\|L_k\|^2_{w}} = \left.\frac{n!}{\Gamma(n+\alpha+1)}\right|_{(\alpha=0)} = \frac{n!}{\Gamma(n+1)} = \frac{n!}{n!} = 1.$$
Vi behöver integraler av typen $I_n = \int_0^\infty x^{\frac{1}{2}} x^n e^{-x} = \int_0^\infty x^{n+\frac{3}{2}-1} e^{-x}\,dx$
$= \Gamma(n+\frac{3}{2}) = (n+\frac{1}{2})\Gamma(n+\frac{1}{2})$:
$$I_0 = \int_0^\infty \sqrt{x}\, e^{-x}\,dx = \{n=0\} = \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{1}{2}\sqrt{\pi}$$
$$I_1 = \int_0^\infty \sqrt{x}\, x e^{-x}\,dx = \{n=1\} = \underbrace{\frac{3}{2}\Gamma(\tfrac{3}{2})} = \frac{3}{2}\cdot\frac{1}{2}\Gamma(\tfrac{1}{2}) = \underbrace{\frac{3}{4}}\sqrt{\pi}.$$
$$I_2 = \int_0^\infty \sqrt{x}\, x^2 e^{-x}\,dx = \{n=2\} = \frac{5}{2}\Gamma(\tfrac{5}{2}) = \frac{5}{2}\cdot\underbrace{\frac{3}{2}\Gamma(\tfrac{3}{2})} = \frac{5}{2}\cdot\underbrace{\frac{3}{4}}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$
$$L_0(x) = 1, \quad L_1(x) = 1-x, \quad L_2(x) = 1-2x+\frac{x^2}{2}$$
$$c_0 = \int_0^\infty \sqrt{x}\, e^{-x}\,dx = I_0 = \frac{1}{2}\sqrt{\pi}$$
$$c_1 = \int_0^\infty \sqrt{x}\,(1-x)e^{-x}\,dx = I_0 - I_1 = \frac{1}{2}\sqrt{\pi} - \frac{3}{4}\sqrt{\pi} = -\frac{1}{4}\sqrt{\pi}.$$
$$c_2 = \int_0^\infty \sqrt{x}\,(1-2x+\frac{x^2}{2})e^{-x}\,dx = I_0 - 2I_1 + \frac{1}{2}I_2 = \left(\frac{1}{2} - 2\frac{3}{4} + \frac{1}{2}\cdot\frac{15}{8}\right)\sqrt{\pi} = -\frac{1}{16}\sqrt{\pi}$$
$$P(x) = \sqrt{\pi}\left[\frac{1}{2}\,1 - \frac{1}{4}(1-x) - \frac{1}{16}\left(1-2x+\frac{x^2}{2}\right)\right] = \underline{\frac{\sqrt{\pi}}{16}\left(3+6x-\frac{1}{2}x^2\right)}.$$

44. Lös Laplaces ekvation $\Delta u = 0$, $0 < a < r < b$ (sfäriska koordinater) med randvillkoren

$$\begin{cases} u = 1 + \cos\theta, & \text{då } r = a \\ u = \cos(2\theta), & \text{då } r = b. \end{cases}$$

E5-44

Lösning. En φ oberoende lösning kan ansättas:

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1}\right) P_n(\cos\theta) \Rightarrow$$

$$\begin{cases} u(a,\theta) = \sum_{n=0}^{\infty} \left(A_n a^n + B_n a^{-n-1}\right) P_n(\cos\theta) = P_0(\cos\theta) + P_1(\cos\theta) \\ u(b,\theta) = \sum_{n=0}^{\infty} \left(A_n b^n + B_n b^{-n-1}\right) P_n(\cos\theta) = \cos 2\theta = 2\cos^2\theta - 1 = C_0 P_0(\cos\theta) + C_2 P_2(\cos\theta) \end{cases}$$

$$= C_0 + C_2 \cdot \frac{1}{2}\left(3\cos^2\theta - 1\right)$$

Identif. av koeff. $\Rightarrow \begin{cases} C_0 - \frac{C_2}{2} = -1 \\ \frac{3}{2} C_2 = 2 \end{cases} \Rightarrow C_2 = \frac{4}{3},\ C_0 = \frac{2}{3} - 1 = -\frac{1}{3}$

Koefficientidentifiering i Fourier-Legendre-serierna ger

$\underline{\underline{n=0:}}$ $\begin{cases} A_0 + \frac{B_0}{a} = 1 \\ A_0 + \frac{B_0}{b} = -\frac{1}{3} \end{cases}$ $\begin{cases} B_0\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{4}{3}, \\ A_0 = 1 - \frac{4}{3}\frac{b}{b-a}, \end{cases}$ $\underline{\underline{B_0 = \frac{4}{3}\frac{ab}{b-a}}}$ $\underline{\underline{A_0 = -\frac{1}{3}\frac{3a+b}{b-a}}}$

$\underline{\underline{n=1:}}$ $\begin{cases} A_1 a + \frac{B_1}{a^2} = 1 \\ A_1 b + \frac{B_1}{b^2} = 0 \end{cases}$ $\begin{cases} A_1 a - A_1 \frac{b^3}{a^2} = 1, \\ B_1 = -A_1 b^3 \end{cases}$ $\underline{\underline{A_1 = -\frac{a^2}{b^3 - a^3}}}$ $\underline{\underline{B_1 = \frac{a^2 b^3}{b^3 - a^3}}}$

$\underline{\underline{n=2:}}$ $\begin{cases} A_2 a^2 + \frac{B_2}{a^3} = 0 \\ A_2 b^2 + \frac{B_2}{b^3} = \frac{4}{3} \end{cases}$ $\begin{cases} B_2 = -A_2 a^5 \\ A_2 b^2 - A_2 \frac{a^5}{b^3} = \frac{4}{3} \end{cases}$ $\underline{\underline{A_2 = \frac{4}{3}\frac{b^3}{b^5 - a^5}}}$ $\underline{\underline{B_2 = -\frac{4}{3}\frac{a^5 b^3}{b^5 - a^5}}}$

$\underline{\underline{n \geq 3:}}$ $\begin{cases} A_n a^n + \frac{B_n}{a^{n+1}} = 0 \\ A_n b^n + \frac{B_n}{b^{n+1}} = 0 \end{cases} \Rightarrow \{\text{för } n \geq 3\},\ \underline{\underline{A_n = B_n = 0}}$

$$u(r,\theta) = \left(A_0 + \frac{B_0}{r}\right) P_0(\cos\theta) + \left(A_1 r + \frac{B_1}{r^2}\right) P_1(\cos\theta) + \left(A_2 r^2 + \frac{B_2}{r^3}\right) P_2(\cos\theta)$$

$$\underline{\underline{= \frac{1}{3}\frac{1}{b-a}\left(\frac{4ab}{r} - 3a - b\right) + \frac{a^2}{b^3 - a^3}\left(\frac{b^3}{r^2} - r\right)\cos\theta + \frac{2}{3}\frac{b^3}{b^5 - a^5}\left(r^2 - \frac{a^5}{r^3}\right)(3\cos^2\theta - 1)}}$$

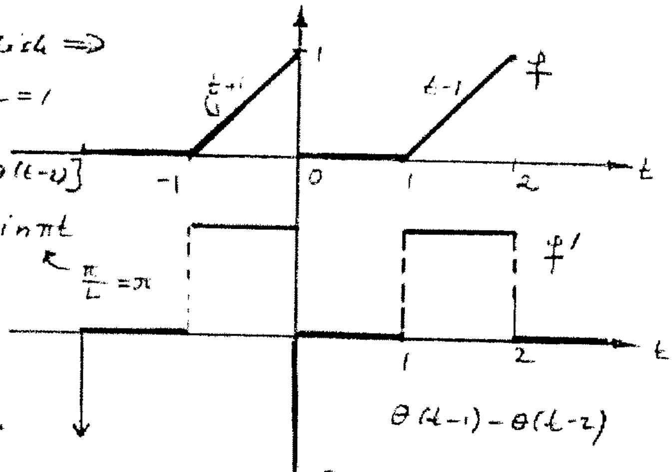
θ betecknar Heaviside-funktionen

48. Bestäm en periodisk lösning till ekvationen $y''-y'+y=f'(t)$, där

$$f(t)=\begin{cases}0 & \text{för} \quad 0<t\le 1\\ t-1 & \text{för} \quad 1<t<2,\end{cases}$$

och f är periodisk med period 2. (Med $f'(t)$ avses distributionsderivatan)

Lösning: f 2-periodisk $\Rightarrow$ $2L=2 \Rightarrow L=1$



$f'(t)=-\delta(t)+[\theta(t-1)-\theta(t-2)]$

Låt $f(t)=\sum_{-\infty}^{\infty} c_n(f)e^{in\pi t}$ $\leftarrow \frac{\pi}{L}=\pi$

$f'(t)=\sum_{-\infty}^{\infty} c_n' e^{in\pi t}$

med $c_n' = in\pi c_n$

$$c_n=\frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\pi t}dt=\frac{1}{2}\int_{-1}^{0}(t+1)e^{-in\pi t}dt$$

$n=0$: $c_0=\frac{1}{2}\int_{-1}^{0}(t+1)\,dt=\frac{1}{2}\left.\frac{(t+1)^2}{2}\right|_{-1}^{0}=\frac{1}{4}$, $c_0'=i(n=0)\pi c_0=0$

$n\neq 0$: $c_n \overset{PI}{=} \frac{1}{2}\left[(t+1)\frac{e^{-in\pi t}}{-in\pi}\right]_{-1}^{0}-\frac{1}{2}\int_{-1}^{0}\frac{e^{-in\pi t}}{-in\pi}dt=\frac{1}{2}\cdot\frac{1}{-in\pi}$

$$-\frac{1}{2}\left[\frac{e^{-in\pi t}}{(-in\pi)^2}\right]_{t=-1}^{0}=-\frac{1}{2}\left(\frac{1}{in\pi}+\frac{1}{(-in\pi)^2}-\frac{(-1)^n}{(-in\pi)^2}\right)$$

$$\frac{1}{in\pi}\cdot\left(-\frac{1}{2}\right)\left(1+\frac{1}{in\pi}-\frac{(-1)^n}{in\pi}\right)$$

$$c_n'=in\pi c_n=-\frac{1}{2}\left(1+\frac{1}{in\pi}-\frac{(-1)^n}{in\pi}\right)=\frac{1}{2in\pi}(-in\pi-1+(-1)^n)$$

Alltså är $c_n'=\frac{1}{2n\pi}\left[-n\pi-i\left((-1)^n-1\right)\right]$, $n\neq 0$; $c_0'=0$.

Sätt $y(t)=\sum_{-\infty}^{\infty} d_n e^{in\pi t}$, $y'(t)=\sum_{n=-\infty}^{\infty} in\pi d_n e^{in\pi t}$, $y''(t)=\sum_{n=-\infty}^{\infty}-n^2\pi^2 d_n e^{in\pi t}$

$$y''-y'+y=f'(t) \Leftrightarrow \sum_{-\infty}^{\infty} d_n\left[1-in\pi-n^2\pi^2\right]e^{in\pi t}=\sum_{n=-\infty}^{\infty} c_n' e^{in\pi t}$$

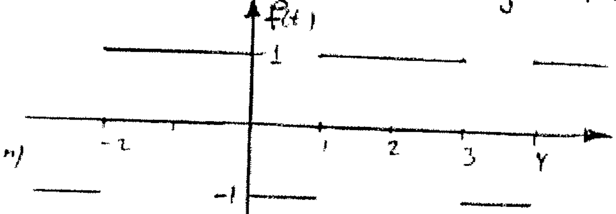
$$\Rightarrow d_n=\frac{c_n'}{1-in\pi-n^2\pi^2} \Rightarrow d_n=\frac{1}{2n\pi}\frac{-n\pi-i\left((-1)^n-1\right)}{1-in\pi-n^2\pi^2}, \quad d_0=0$$

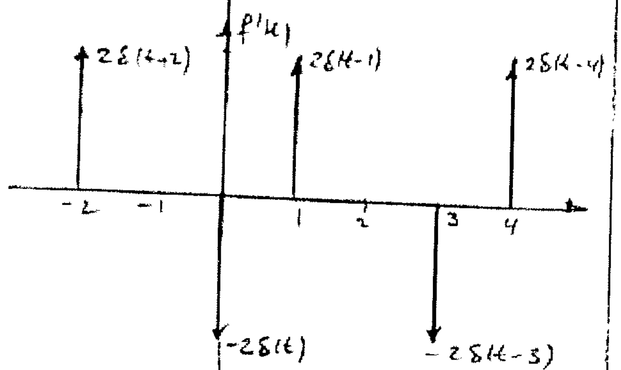
$$y_p=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{2n\pi}\cdot\frac{n\pi-i\left(1-(-1)^n\right)}{n^2\pi^2-1+in\pi}e^{in\pi t}$$

49. Om funktionen f gäller att

$$f(t)=\begin{cases}-1 & \text{för } 0<t<1\\ 1 & \text{för } 1<t<3,\end{cases}$$

och $f(t)$ är periodisk med perioden 3. Bestäm $f'(t)$ (distributionsderivatan) och utveckla $f'(t)$ i komplex trigonometrisk Fourierserie. Använd resultatet för att bestämma Fourierserieutvecklingen av $f(t)$.





Lösning: Fig. visar att

$$f'(t)=\sum_{n=-\infty}^{\infty}2\delta(t-3n-1)-\sum_{n=-\infty}^{\infty}2\delta(t-3n)$$

$$=2\sum_{n=-\infty}^{\infty}\left[\delta(t-3n-1)-\delta(t-3n)\right]$$

$T=3,\quad \Omega=\frac{2\pi}{T}=\frac{2\pi}{3}$

$$f'(t)=\sum_{n=-\infty}^{\infty}c_n e^{in\frac{2\pi}{3}t},\ \text{där}$$

$$\underline{\underline{c_n}}=\frac{1}{3}\int_{-1}^{2}f'(t)\,e^{-in\frac{2\pi}{3}t}\,dt=$$

$$=\frac{1}{3}\int_{-1}^{2}\left[2\delta(t-1)-2\delta(t)\right]e^{-in\frac{2\pi}{3}t}\,dt=\frac{2}{3}\left(e^{-in\frac{2\pi}{3}}-e^{0}\right)=\underline{\underline{\frac{2}{3}\left(e^{-in\frac{2\pi}{3}}-1\right)}},$$

$$c_0=0,\quad \underline{\underline{f'(t)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}c_n e^{in\frac{2\pi}{3}t}}}$$

$$f(t)=\sum_{n\neq 0}\frac{c_n}{in\frac{2\pi}{3}}e^{in\frac{2\pi}{3}t}+A.$$

A är Fourierkoeff för $n=0$ för $f(t)$, varför $A=\frac{1}{3}\int_0^3 f(t)\,dt=$

$$=\frac{1}{3}(-1+2)=\frac{1}{3}$$

$$\underline{\underline{f(t)=\frac{1}{3}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{-in\frac{2\pi}{3}}-1}{in\pi}e^{in\frac{2\pi}{3}t}}}.$$

56. Bestäm lösningen $f(t)$, $t>0$, ur integralekvationen

$$\begin{cases} f''(t) - 4f'(t) + f(t) + 6\int_0^t f(\tau)\,d\tau = 2e^t \\ f(0-) = 1, \quad f'(0) = 0 \end{cases}$$

Lösning: $\mathcal{L}$-transform $\Rightarrow$

$$s^2F(s) - s\overbrace{f(0-)}^{=1} - \overbrace{f'(0-)}^{=0} - 4sF(s) + 4\overbrace{f(0-)}^{=1} + F(s) + 6\frac{F(s)}{s} = \frac{2}{s-1}$$

$$\left(s^2 - 4s + 1 + \frac{6}{s}\right)F(s) = s - 4 + \frac{2}{s-1},$$

$$F(s) = \frac{s-4+\frac{2}{s-1}}{s^2-4s+1+\frac{6}{s}} = \frac{s}{s-1}\cdot\frac{2+(s-1)(s-4)}{s^3-4s^2+s+6} = \{s=-1,2,3 \text{ är rötterna till nämnaren}\} = \frac{s}{s-1}\cdot\frac{s^2-5s+6}{(s+1)(s^2-5s+6)} = \frac{s}{(s-1)(s+1)}$$

$$= \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}.$$

$$f(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t) \quad \text{för } t \geq 0.$$