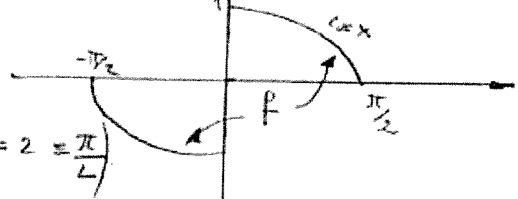
3. Utveckla funktionen $\cos(x)$ i sinusserie på intervallet $(0, \frac{\pi}{2})$.
Använd resultatet för att beräkna
$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}.$$

Lösning: Utv. $\cos(x)$ till en $2L = \pi$ periodisk udda funktion f på $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\Omega x + b_n \sin n\Omega x\right)$$



f udda $\Rightarrow a_n = 0, \ \forall n \geq 0$

(obs! $\Omega = \frac{2\pi}{T} = \{\text{perioden } T = \pi\} = 2 = \frac{\pi}{L}$)

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx = \left\{L = \frac{\pi}{2}\right\} = \frac{4}{\pi}\int_0^{\pi/2} \cos(x) \sin\frac{n\pi x}{\pi/2}\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx\,dx = \frac{4}{\pi}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\sin(2n+1)x + \sin(2n-1)x\right]dx$$

$$= \frac{2}{\pi}\left(\frac{-1}{2n+1}\cos(2n+1)x\Big|_0^{\pi/2} - \frac{1}{2n-1}\cos(2n-1)x\Big|_0^{\pi/2}\right) = \frac{2}{\pi}\left(\frac{1}{2n+1} + \frac{1}{2n-1}\right) = \frac{8n}{\pi(4n^2-1)}$$

$$\therefore \quad f(x) \cong \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1}\sin(2nx) \quad \text{dvs}$$

$$(1) \qquad \cos x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin(2nx), \qquad \text{för} \quad x \in (0, \pi/2).$$

Multipl. (1) med $\cos(x)$, $\int_0^{\pi/2}$ & byt! $\int_0^{\pi/2}$ & $\sum$ i HL $\Rightarrow$

$$\int_0^{\pi/2} \cos x \cos x\,dx = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\int_0^{\pi/2} \sin(2nx)\cos(x)\,dx \iff$$

$$\int_0^{\pi/2}\left[\frac{1}{2} + \frac{1}{2}\cos(2x)\right]dx = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\cdot\frac{2n}{4n^2-1} \iff$$

$$\frac{\pi}{4} = \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2} \Rightarrow \sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}.$$

4. Låt $f(t) = 1 - t^2$ för $|t| \le 1$ och låt f vara 2-periodisk. Bestäm en begränsad lösning till

$$\text{(PDE)} \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x>0, \quad -\infty<t<\infty \\ u(0,t) = f(t), & -\infty<t<\infty \end{cases} \text{(RV)}$$

Lösning: Utveckla $f(t)$ i trigonometrisk Fourierserie $f(t) = \sum_{-\infty}^{\infty} c_n e^{in\Omega t}$

där $\Omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$; $c_n = \frac{1}{2}\int_{-1}^{1}(1-t^2)e^{-in\pi t}dt = [n \neq 0] = \frac{1}{2}\left[(1-t^2)\frac{e^{-in\pi t}}{-in\pi}\right]_{-1}^{1}$

$-\frac{1}{2}\int_{-1}^{1}(-2t)\frac{e^{-in\pi t}}{-in\pi}dt = \left[t\frac{e^{-in\pi t}}{(-in\pi)^2}\right]_{-1}^{1} - \int_{-1}^{1}\frac{e^{-in\pi t}}{(-in\pi)^2}dt = \frac{-1}{n^2\pi^2}\left(e^{-in\pi}+e^{in\pi}\right) - \underbrace{\left[\frac{e^{-in\pi t}}{(-in\pi)^3}\right]_{-1}^{1}}_{=0}$

$= \underline{\frac{-2(-1)^n}{n^2\pi^2}}$, $\quad c_0 = \int_0^1(1-t^2)dt = 1-\frac{1}{3} = \frac{2}{3}$

$$\therefore f(t) = \frac{2}{3} - \frac{2}{\pi^2}\sum_{\substack{-\infty \\ n\neq 0}}^{\infty}\frac{(-1)^n}{n^2}e^{in\pi t}$$

Ansätt en periodisk lösning. $u(x,t) = \sum_{-\infty}^{\infty} u_n(x)e^{in\pi t}$, där $u_n(x) = \frac{1}{2}\int_{-1}^{1}u(x,t)e^{-in\pi t}$ är begr.

$u(0,t) = \sum_{-\infty}^{\infty} u_n(0)e^{in\pi t} = f(t) = \sum_{-\infty}^{\infty} c_n e^{in\pi t}$, varför $\underline{u_n(0) = c_n}$ för alla n.

$$\text{(PDE)} \iff \sum_{-\infty}^{\infty} u_n(x)(in\pi)e^{in\pi t} = \sum_{-\infty}^{\infty} u_n''(x)e^{in\pi t} \implies \begin{cases} (in\pi)u_n - u_n'' = 0 & \text{(ODE)} \\ u_n(0) = c_n, & \forall n. \end{cases}$$

kar. ekv. $r^2 = in\pi \implies \begin{cases} r_{1,2} = \pm\frac{1}{\sqrt{2}}(1+i)\sqrt{n\pi} & n>0 \\ r_{1,2} = \pm\frac{1}{\sqrt{2}}(1-i)\sqrt{|n|\pi} & n<0, \end{cases}$

(ty $\sqrt{\pm i} = \frac{1}{\sqrt{2}}(1\pm i) \iff \pm i = \frac{1}{2}(1\pm i)^2 \iff \pm i = \frac{1}{2}(1+i^2 \pm 2i) \iff \pm i = \pm i$)

rötterna är r_1 och r_2 där $\mathrm{Re}\, r_1 < 0$, $\mathrm{Re}\, r_2 > 0$ $\underline{\text{, allmänna lösningen}}$

$u_n(x) = A_n e^{r_1 x} + B_n e^{r_2 x}$ Men $|e^{r_2 x}| \to \infty$, då $x \to \infty$, varför $B_n = 0$. (u_n begr.)

för $n = 0$ får istället $u_0 = A_0 + B_0 x$ ($u_0'' = 0$ i $\underline{\text{(ODE)}}$).

där $B_0 = 0$ för begränsad lösning.

$$\implies \begin{cases} u_n(x) = A_n e^{-\frac{1}{\sqrt{2}}(1+i\,\mathrm{sign}\, n)\sqrt{|n|\pi}\,x} \\ u_n(0) = A_n = c_n, \quad n\in\mathbb{Z} \end{cases} \implies \underline{u_n(x) = c_n e^{-\sqrt{\frac{|n|\pi}{2}}(1+i\,\mathrm{sign}\, n)x}}$$

$$u(x,t) = \sum_{-\infty}^{\infty} c_n e^{-\sqrt{\frac{|n|\pi}{2}}(1+i\,\mathrm{sign}\, n)x} e^{in\pi t}$$

$$= c_0 + \sum_{n=1}^{\infty}\left\{c_n e^{-\sqrt{\frac{n\pi}{2}}} e^{i\left(n\pi t - \sqrt{\frac{n\pi}{2}}x\right)} + c_{-n}e^{-\sqrt{\frac{n\pi}{2}}} e^{i\left(-n\pi t + \sqrt{\frac{n\pi}{2}}x\right)}\right\}$$

$$\therefore u(x,t) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{n-1}}{n^2\pi^2}e^{-\sqrt{\frac{n\pi}{2}}}\cos\left(n\pi t - \sqrt{\frac{n\pi}{2}}x\right).$$

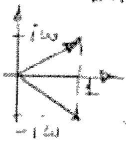
8. Funktionen $f(t)$ har Fouriertransformen $\frac{1-i\omega}{1+i\omega}\frac{\sin\omega}{\omega}$. Beräkna

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt$$

<u>Lösning</u>: $\hat{f}(\omega)=\frac{1-i\omega}{1+i\omega}\cdot\frac{\sin\omega}{\omega}$. Enligt Parsevals

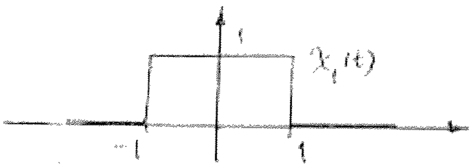
$$\int_{-\infty}^{\infty}|f(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1-i\omega}{1+i\omega}\right|^2\frac{\sin^2\omega}{\omega^2}\,d\omega$$

$$=\left\{\left|\frac{1-i\omega}{1+i\omega}\right|=1\right.$$



$$\left.\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}\right)^2 d\omega=$$

Vi har att $x_1(t)=\theta(t+1)-\theta(t-1)\xrightarrow{F}2\frac{\sin\omega}{\omega}$.



Alltså

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}\right)^2 d\omega=\int_{-\infty}^{\infty}\left\{\frac{1}{2}\left[\theta(t+1)-\theta(t-1)\right]\right\}^2 dt=\frac{1}{4}\int_{-1}^{1}dt=\frac{1}{2}.$$

$$\therefore\ \underline{\underline{\int_{-\infty}^{\infty}|f(t)|^2\,dt=\frac{1}{2}.}}\quad\boxtimes$$

11. Ange Fouriertransformen till funktionen

$$f(t)=\int_0^2 \frac{\sqrt{\omega}}{1+\omega}\, e^{i\omega t}\, d\omega .$$

Beräkna a) $\int_{-\infty}^{\infty} f(t)\cos t \, dt$, b) $\int_{-\infty}^{\infty} |f(t)|^2 \, dt$.

Lösning:

$$f(t)=\int_0^2 \frac{\sqrt{\omega}}{1+\omega}\, e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$$

varav vi använder

$$\hat{f}(\omega)=\begin{cases} \dfrac{2\pi\sqrt{\omega}}{1+\omega}, & \text{om } 0<\omega<2 \\ 0 & \text{för övrigt} \end{cases}$$

a) $\hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \implies$

$$\int_{-\infty}^{\infty} f(t)\cos t\, dt = \int_{-\infty}^{\infty} f(t)\,\tfrac{1}{2}\left(e^{it}+e^{-it}\right) dt = \tfrac{1}{2}\left[\hat{f}(-1)+\hat{f}(1)\right] = \tfrac{1}{2}\left(0+\tfrac{2\pi}{2}\right)=\frac{\pi}{2}$$

b) Parseval.

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_0^2 \frac{(2\pi)^2\omega}{(1+\omega)^2}\, d\omega$$

$$=[1+\omega=y] = 2\pi\int_1^3 \frac{y-1}{y^2}\, dy = 2\pi\int_1^3\left(\frac{1}{y}-\frac{1}{y^2}\right) dy = 2\pi\left[\ln y+\frac{1}{y}\right]_1^3$$

$$=2\pi\left(\ln 3+\tfrac{1}{3}-0-1\right) = 2\pi\left(\ln 3-\tfrac{2}{3}\right).$$

a. $\underline{\underline{\int_{-\infty}^{\infty} f(t)\cos t\, dt = \frac{\pi}{2}.}}$ b. $\underline{\underline{\int_{-\infty}^{\infty} |f(t)|^2\, dt = 2\pi\left(\ln 3-\frac{2}{3}\right).}}$

15. Bestäm en lösning till ekvationen

$$u(t) + \int_{-\infty}^{t} e^{\tau - t}\, u(\tau)\, d\tau = e^{-2|t|}.$$

Lösning. Vi har att $\displaystyle\int_{-\infty}^{t} e^{\tau-t}\, u(\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-(t-\tau)}\, \theta(t-\tau)\, u(\tau)\, d\tau$

$$= \{e^{-t}\theta(t)\} * \{u(t)\}.$$

F-transf $\Rightarrow$ $\displaystyle \hat{u}(\xi) + \frac{1}{1+i\xi}\hat{u}(\xi) = \frac{4}{\xi^2+4} \Rightarrow \hat{u}(\xi)\left(1+\frac{1}{i\xi+1}\right) = \frac{4}{\xi^2+4}$

$$\hat{u}(\xi) = \frac{4(i\xi+1)}{(i\xi+2)(\xi^2+4)} = 4\cdot\left(\frac{i\xi+2}{(i\xi+2)(\xi^2+4)} - \frac{1}{(i\xi+2)(\xi^2+4)}\right).$$

[PBU] $\Rightarrow$ $\displaystyle \frac{1}{(i\xi+2)(\xi^2+4)} = \frac{1}{(2-i\xi)(2+i\xi)^2} = \frac{A}{2-i\xi} + \frac{B}{2+i\xi} + \frac{C}{(2+i\xi)^2}$

$\displaystyle\left\{\text{HP} \Rightarrow A = \frac{1}{16},\quad C = \frac{1}{4}\right\} = \left[\frac{1}{16}(2+i\xi)^2 + B(4+\xi^2) + \frac{1}{4}(2-i\xi)\right] \Big/ N$

$\{$där N = Nämnaren $= (i\xi+2)(\xi^2+4)\}$.

Identif. av koeff. i ξ^2: $\displaystyle -\frac{1}{16} + B = 0 \Rightarrow B = \frac{1}{16}$

Alltså $\displaystyle \hat{u}(\xi) = \frac{4}{\xi^2+4} \underbrace{- \frac{1}{4}\frac{1}{2-i\xi} - \frac{1}{4}\frac{1}{2+i\xi}} - \frac{1}{(2+i\xi)^2}$

$$= \frac{4}{\xi^2+4} - \frac{1}{4}\cdot\frac{4}{\xi^2+4} + \frac{d}{d\xi}\left(\frac{i}{2+i\xi}\right)$$

$$\Rightarrow \quad u(t) = \frac{3}{4}e^{-2|t|} - t\, e^{-2t}\,\theta(t).$$

Alt. 1
Eö-25

25. Lös problemet
$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = y, & 0<x<2,\ 0<y<1 \\ u(x,0)=0, & u(x,1)=0 \\ u(0,y)=y-y^3, & u(2,y)=0 \end{cases} \quad \text{(BV)}$$

Alternativ 1

Lösning [Se också Föreläsning anteck. Om Homogenisering]:

Bestäm $w(y)$ så att $w''(y)=y$, $w(0)=w(1)=0$. $w'(y)=\frac{1}{2}y^2+A$.

$w(y)=\frac{1}{6}y^3+Ay+B$, $w(0)=0 \Rightarrow B=0$, $w(1)=0 \Rightarrow \frac{1}{6}+A=0 \Rightarrow A=-\frac{1}{6}$.

Men får $w(y)=\frac{1}{6}(y^3-y)$.

Sätt $v(x,y)=u(x,y)-w(y)$. Då satisfierar v
$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \\ v(x,0)=v(x,1)=0 \\ v(0,y)=u(0,y)-w(y)= y-y^3-\frac{1}{6}(y^3-y)=\frac{7}{6}(y-y^3) \\ v(2,y)=u(2,y)-w(y)=\frac{1}{6}(y-y^3). \end{cases}$$

Använd variabelseparation $V(x,y)=X(x)Y(y)\neq 0$ (så att de homogena villkoren uppfylls)

$\Rightarrow X''Y+XY''=0$; $\dfrac{X''}{X}=-\dfrac{Y''}{Y}=\lambda \Rightarrow$ (I) $\begin{cases} -Y''=\lambda Y \\ Y(0)=Y(1)=0 \end{cases}$ (II) $X''=\lambda X$

S-L-problemet (I) har lösn. $\underline{\underline{\lambda=\lambda_n=(n\pi)^2}}$, $\underline{\underline{Y=Y_n(y)=\sin n\pi y}}$, $n\geq 1$.

Allmänna lösn. till (II) med $\lambda=\lambda_n$ kan skrivas, $X_n(x)=A_n\sinh n\pi x+B_n\sinh n\pi(2-x)$.

Ansätt totallösningen $v(x,y)=\sum_{n=1}^{\infty}[A_n\sinh n\pi x+B_n\sinh n\pi(2-x)]\sin n\pi y$.

Vi vill ha: $v(0,y)=\sum_{n=1}^{\infty} B_n \sinh 2n\pi\cdot\sin n\pi y=\frac{7}{6}(y-y^3)$.

$v(2,y)=\sum_{n=1}^{\infty} A_n\sinh 2n\pi\cdot \sin n\pi y=\frac{1}{6}(y-y^3)$.

Utveckla $\frac{1}{6}(y-y^3)$ i Fourierserie m.a.p. det fullständiga ON-systemet $\{\sin n\pi y\}_{n=1}^{\infty}$:

$\frac{1}{6}(y-y^3)=\sum_{n=1}^{\infty} C_n\sin n\pi y$, där $C_n=\frac{1}{1/2}\int_0^1 \frac{1}{6}(y-y^3)\sin n\pi y\,dy$. Då blir

$A_n\sinh 2n\pi = C_n$ och $B_n=7A_n$

$C_n=\frac{1}{3}\left[(y-y^3)\cdot\frac{-\cos n\pi y}{n\pi}\right]_0^1+\frac{1}{3}\int_0^1(1-3y^2)\frac{\cos n\pi y}{n\pi}dy=\frac{1}{3}\left[(1-3y^2)\frac{\sin n\pi y}{(n\pi)^2}\right]_0^1-$

$-\frac{1}{3}\int_0^1(-6y)\frac{\sin n\pi y}{(n\pi)^2}dy=-2\left[y\frac{\cos n\pi y}{(n\pi)^3}\right]_0^1+2\int_0^1\frac{\cos n\pi y}{(n\pi)^3}dy=-\frac{2\cos n\pi}{(n\pi)^3}=\frac{2(-1)^{n-1}}{(n\pi)^3}$,

$A_n=\dfrac{2(-1)^{n-1}}{(n\pi)^3\sinh(2n\pi)}$, $B_n=\dfrac{14(-1)^{n-1}}{(n\pi)^3\sinh(2n\pi)}$

$$u(x,y)=\frac{1}{6}(y^3-y)+\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\sinh 2n\pi}[\sinh n\pi x+7\sinh n\pi(2-x)]\sin n\pi y.$$

Alt.

Alternativ

EÖ-25

Via Multiple F-serie

Alt. 2

25. Alternativ genom F-serie i produkt form: $(BV)\begin{cases} u_{xx}+u_{yy}=y & 0<x<2,\ 0<y<1 \\ u(x,0)=u(x,1)=0 \\ u(0,y)=y-y^3, & u(2,y)=0\end{cases}$

Lösning: Sätt $v(x,y)=u(x,y)-\varphi(x,y)$ där att φ polynom i x & y med

$\varphi_{xx}=0$ & $\varphi_{yy}=Ay+B$ och φ satisf. (BV) $\Rightarrow \varphi(x,y)=(ax+b)(py^3+qy^2+ry+s)$

$\varphi(x,0)=(ax+b)s=0 \quad \forall x \Rightarrow \underline{s=0}$, $\varphi(x,1)=(ax+b)(p+q+r)=0,\ \forall x \Rightarrow \underline{p+q+r=0}$

$\varphi(0,y)=b(py^3+qy^2+ry)=y-y^3 \Rightarrow p=-\frac{1}{b},\ q=0,\ r=\frac{1}{b}$

$\varphi(2,y)=(2a+b)(-\frac{1}{b}y^3+\frac{1}{b}y)=(\frac{2a}{b}+1)(-y^3+y)=0 \overset{\forall y}{\Longrightarrow} b=-2a$

Alltså $\underline{\varphi(x,y)}=(ax+b)(py^3+qy^2+ry+s)=\underset{-2a}{\frac{1}{b}}(ax-2a)(-y^3+y)=\underline{\frac{1}{2}(2-x)(y-y^3)}$

$v_{xx}+v_{yy}=u_{xx}+u_{yy}-(\varphi_{xx}+\varphi_{yy})=y-(0+\frac{1}{2}(2-x)(-6y))=y-\frac{1}{2}(2-x)(-6y)=\underline{7y-3xy}$

Ekv. för $v(x,y)$:

$\boxed{\text{Jfr. med Multiple F-serie Folland s.d 125. } \ell_1=2,\ \ell_2=1}$ $\begin{cases} v_{xx}+v_{yy}=7y-3xy=y(7-3x) \\ v(x,0)=v(1,0)=0 \\ v(0,y)=v(2,y)=0\end{cases}$

Var. Sep. $\Rightarrow v(x,y)=X(x)Y(y)\neq 0$ för homogen ekv. gäller $\frac{X''}{X}=-\frac{Y''}{Y}=\lambda$

(X) $\begin{cases} X''+\lambda X=0 \\ X(0)=X(2)=0\end{cases} \Rightarrow \lambda_m=\left(\frac{m\pi}{2}\right)^2,\ X_m(x)=\sin\frac{m\pi}{2}x,\ m\geq 1$

(Y) $\begin{cases} Y''-\lambda Y=0 \\ Y(0)=Y(1)=1\end{cases} \Rightarrow Y_n(y)=a_n\sin n\pi y,\ n\geq 1.$

Teknik II, Homogen BV $\Rightarrow$ Använd multip. Fourier Sinus serie med:

$v(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\frac{m\pi x}{2}\sin n\pi y \Rightarrow$

$v_{xx}+v_{yy}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\underbrace{a_{mn}\left(-\frac{m^2\pi^2}{4}-n^2\pi^2\right)}_{!\,=\,d_{mn}}\sin\frac{m\pi x}{2}\sin n\pi y=y(7-3x),$

Alltså d_{mn} är F-sinus koeff. för $y(7-3x)$ m.a.p. multiple bas $\sin\frac{m\pi}{2}x\sin n\pi y$

dvs. $d_{mn}=\frac{4}{1\times 2}\int_0^1 y\sin n\pi y\,dy\int_0^2(7-3x)\sin\frac{m\pi x}{2}\,dx=$

$2\left[y\frac{-\cos n\pi y}{n\pi}\Big|_0^1-\int_0^1\overset{\to 0}{\frac{-\cos n\pi y}{n\pi}}dy\right]\cdot\left[(7-3x)\frac{-2}{m\pi}\cos\frac{m\pi x}{2}\Big|_0^2-\int_0^2(-3)\frac{-2}{m\pi}\overset{\to 0}{\cos\frac{m\pi x}{2}dx}\right]$

$=-\frac{2}{n\pi}\cos n\pi\cdot\frac{-2}{m\pi}\left((7-3(2))\cos m\pi-7\right)=\frac{4(-1)^n}{nm\pi^2}\left((-1)^m-7\right)\equiv a_{mn}\left(-\frac{m^2\pi^2}{4}-n^2\pi^2\right)$

$\Rightarrow a_{mn}=\frac{4(-1)^n(7-(-1)^m)}{nm\left(\frac{m^2}{4}+n^2\right)\pi^4}$, $v(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\frac{m\pi x}{2}\sin n\pi y$

$\underline{\underline{u(x,y)=\frac{1}{2}(2-x)(y-y^3)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\frac{m\pi x}{2}\sin n\pi y.}}$

EÖ-4

45. Sök en begränsad lösning till

$$\begin{cases} u_t = k\,u_{xx}, & -\infty<x<\infty \quad t>0 \\ u(x,0) = (1-2x^2)e^{-x^2}, & -\infty<x<\infty. \end{cases}$$

<u>Lösning</u>: $\mathcal{F}$-transf. i x-led $\Rightarrow$ $\hat{u}_t = k(i\xi)^2\hat{u}$, $\hat{u}_t = -k\xi^2\hat{u}$,

$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{-k\xi^2 t}$; där $\hat{u}(\xi,0) \overset{\mathcal{F}}{\leftarrow} (1-2x^2)e^{-x^2} := u_0(x)$

$$\mathcal{F}\left[e^{-ax^2/2}\right](\xi) = \sqrt{\frac{2\pi}{a}}\, e^{-\xi^2/2a} \Longrightarrow \{a=2\} \Rightarrow \mathcal{F}\left[e^{-x^2}\right](\xi) = \sqrt{\pi}\, e^{-\xi^2/4}$$

$$\mathcal{F}\left[-x^2e^{-x^2}\right](\xi) = \mathcal{F}\left[(-ix)^2e^{-x^2}\right](\xi) = \left(\frac{d}{d\xi}\right)^2\mathcal{F}\left[e^{-x^2}\right](\xi) =$$

$$= \sqrt{\pi}\left(e^{-\xi^2/4}\right)'' = \sqrt{\pi}\left(-\frac{\xi}{2}e^{-\xi^2/4}\right)' = \sqrt{\pi}\left(\frac{\xi^2}{4}-\frac{1}{2}\right)e^{-\xi^2/4},$$

$$\therefore\ u_0(x) = (1-2x^2)e^{-x^2} \overset{\mathcal{F}}{\supset} \cancel{\sqrt{\pi}\, e^{-\xi^2/4}} + \sqrt{\pi}\,\frac{\xi^2}{2}e^{-\xi^2/4} - \cancel{\sqrt{\pi}\, e^{-\xi^2/4}} = \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2/4}$$

$$u(\xi,t) = \frac{\sqrt{\pi}}{2}\xi^2e^{-\xi^2/4}\cdot e^{-k\xi^2 t} = \frac{\sqrt{\pi}}{2}\xi^2 e^{-(kt+\frac{1}{4})\xi^2} \overset{\text{sätt}}{=} \left\{\frac{1}{2a} = kt+\frac{1}{4}\right\}$$

$$= \frac{\sqrt{\pi}}{2}\xi^2e^{-\xi^2/2a} = \frac{1}{2}\sqrt{\frac{a}{2}}\,\xi^2\sqrt{\frac{2\pi}{a}}\,e^{-\xi^2/2} \overset{\mathcal{F}}{\leftarrow} \frac{1}{2}\sqrt{\frac{a}{2}}\left(i\frac{d}{dx}\right)^2\left(e^{-ax^2/2}\right)$$

$$= \frac{-1}{2}\sqrt{\frac{a}{2}}\left(-axe^{-ax^2/2}\right)' = -\frac{1}{2}\sqrt{\frac{a}{2}}\left(-a+a^2x^2\right)e^{-ax^2/2}$$

$$= \left(\frac{a}{2}\right)^{3/2}(1-ax^2)e^{-ax^2/2} = \left\{a = \frac{2}{4kt+1}\right\}$$

$$= \left(\frac{1}{4kt+1}\right)^{3/2}\left(1-\frac{2x^2}{4kt+1}\right)e^{-x^2/(4kt+1)}$$

$$\therefore\ \underline{\underline{u(x,t) = \frac{1}{(4kt+1)^{5/2}}\left(4kt+1-2x^2\right)e^{-\frac{x^2}{4kt+1}}}}.$$

EÖ-47

47. Låt f tillhöra $L^2(\mathbb{R})$ och sök en lösning till

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & -\infty < x < \infty, \quad 0 < y < a \\ u(x,0) = 0, & u(x,a) = f(x). \end{cases}$$

Visa att

$$\int_{-\infty}^{\infty} |u(x,y)|^2 dx \le \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Lösning: F-transf. i x-led; $\hat{u}(\xi,y) = F_x[u(x,y)] \Rightarrow$

$$(i\xi)^2 \hat{u} + \frac{\partial^2 \hat{u}}{\partial y^2} = 0, \quad \hat{u}_{yy} - \xi^2 \hat{u} = 0; \quad \hat{u}(\xi,y) = C_1(\xi)\sinh(\xi y) + C_2(\xi)\cosh(\xi y)$$

$$\hat{u}(\xi,0) = C_2(\xi) = 0, \quad \hat{u}(\xi,a) = C_1(\xi)\sinh(\xi a) = \hat{f}(\xi)$$

$$C_1(\xi) = \frac{\hat{f}(\xi)}{\sinh(\xi a)}, \quad \hat{u}(\xi,y) = \frac{\sinh(\xi y)}{\sinh(\xi a)}\hat{f}(\xi)$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh(\xi y)}{\sinh(\xi a)} \hat{f}(\xi) e^{i\xi x} d\xi$$

Ald. Enl Beta (se också lösning till 7.3:4) är

$$F_t\left[\frac{\sinh at}{\sinh bt}\right](\omega) = \frac{\pi \sin \frac{\pi a}{b}}{b\cosh \frac{\omega a}{b} + b\cos\frac{\pi a}{b}} \quad \text{för } 0 < a < b,$$

Sätt $a = y$, $b = a$ och använd symmetriregeln, så får

$$F\left[\frac{1}{2a} \frac{\sin \frac{\pi y}{a}}{\cosh \frac{\pi x}{a} + \cos\frac{\pi y}{a}}\right] = \frac{\sinh \xi y}{\sinh \xi a}$$

Så att

$$u(x,y) = \frac{1}{2a}\int_{-\infty}^{\infty} \frac{\sin\frac{\pi y}{a}}{\cosh\frac{\pi(x-t)}{a} + \cos\frac{\pi y}{a}} f(t)\,dt$$

$$\int_{-\infty}^{\infty} |u(x,y)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(\xi,y)|^2 d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left(\frac{\sinh \xi y}{\sinh \xi a}\right)^2}_{\le 1} |\hat{f}(\xi)|^2 d\xi$$

$$\le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |f(x)|^2 dx \quad \text{enl Plancherel.}$$

57. Låt $u(x,t)$ vara lösningen till begynnelsevärdesproblemet

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad t>0, \quad 0<x<\pi & (DE) \\ u(0,t) = u(\pi,t) = 0 & (RV) \\ u(x,0) = 0, \quad u_t(x,0) = g(x). & (BV) \end{cases}$$

Bevisa att för $t>0$

$$\int_0^{\pi} |u_t(x,t)|^2 dx \le \int_0^{\pi} |g(x)|^2 dx$$

Lösning: Variabelseparation, sätt $u(x,t) = X(x)T(t) \neq 0$ i (DE);

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \lambda = -\mu^2, \ \mu>0 \quad (\lambda=0 \wedge \lambda>0 \text{ ger trivial lösn.})$$

$$(\mathrm{I}) \begin{cases} X'' + \mu^2 X = 0 \\ X(0) = X(\pi) = 0 \end{cases} \qquad (\mathrm{II}) \begin{cases} T'' + c^2\mu^2 T = 0 \\ T(0) = 0, \quad T'(0) = g(x) \end{cases}$$

(I) är S-L-problem med lösningar: $X(x) = A\sin\mu x + B\cos\mu x$

$X(0) = B = 0, \quad X(\pi) = A\sin(\mu\pi) = 0 \overset{A\neq 0}{\Longrightarrow} \mu = 1,2,\ldots, (\mu = n > 0)$

egenpar: $\underline{\underline{\lambda_n = -n^2; \quad X_n(x) = \sin(nx), \quad n \ge 1}}$.

p.s.s. $T_n(t) = P_n \sin(nct) + q_n \cos(nct), \quad T_n(0) = q_n = 0 \Rightarrow$

$\underline{\underline{T_n(t) = P_n \sin(nct)}}$.

Superposition ger $u(x,t) = \sum_{n=1}^{\infty} P_n \sin(nct)\sin(nx)$.

$u_t(x,t) = \sum_{n=1}^{\infty} ncP_n \cos nct \sin(nx) \Rightarrow u_t(x,0) = \sum_{n=1}^{\infty} cnP_n \sin nx = g(x)$

$$\underline{\underline{\int_0^{\pi} |u_t(x,t)|^2 dx}} \le \sum_{n=1}^{\infty} |cnP_n|^2 \cos^2(nct) \cdot \frac{\pi}{2} \le \sum_{n=1}^{\infty} (cnP_n)^2 \frac{\pi}{2}$$

$$= \int_0^{\pi} |u_t(x,0)|^2 dx = \underline{\underline{\int_0^{\pi} |g(x)|^2 dx}}.$$