Fourieranalys MVE030 och Fourier Metoder MVE290 23.augusti.2016
Betygsgränser: 3: 40 poäng, 4: 50 poäng, 5: 60 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA och en typgodkänd räknedosa.
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1. (Prove properties of solutions to regular Sturm-Lioville problems) Låt f och g vara egenfunktioner till ett regulärt SLP i intervallet $[a, b]$ med $w \equiv 1$. Låt λ vara egenvärden till f och μ vara dess till g. Bevisa:

   (a) $\lambda \in \mathbb{R}$ och $\mu \in \mathbb{R}$;

   (b) Om $\lambda \neq \mu$, gäller:
   $$\int_a^b f(x)\overline{g(x)}dx = 0.$$

   By definition we have $Lf + \lambda f = 0$. Moreover, L is self-adjoint, so we have
   $$\langle Lf, f\rangle = \langle f, Lf\rangle.$$

   By definition,
   $$\langle Lf, f\rangle = \int_a^b L(f)(x)\overline{f(x)}dx.$$

   Thus, we have
   $$-\lambda \int_a^b |f(x)|^2 dx = -\overline{\lambda} \int_a^b |f(x)|^2 dx \iff \lambda = \overline{\lambda}.$$

   The last statement holds because
   $$\int_a^b |f(x)|^2 dx = ||f||_{\mathcal{L}^2} = 0 \iff f \equiv 0,$$

   and by definition of f as an eigenfunction, $f \not\equiv 0$. Same proof holds for μ.

   For the second part, we use basically the same argument based on self-adjointness:
   $$\langle Lf, g\rangle = \langle f, Lg\rangle.$$

   By assumption
   $$\langle Lf, g\rangle = -\lambda\langle f, g\rangle = \langle f, Lg\rangle = \langle f, -\mu g\rangle = -\overline{\mu}\langle f, g\rangle = -\mu\langle f, g\rangle.$$

Thus, if $\langle f, g\rangle \neq 0$, this forces $\lambda = \mu$, which is false. Hence, the only viable option is that $\langle f, g\rangle = 0$. By definition,

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

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2. (Prove the best approximation theorem) Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara en ortogonal mängd i ett Hilbert-rum, H. Om $f \in H$, bevisa:

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle\phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n\phi_n||, \quad \forall\{c_n\}_{n\in\mathbb{N}} \in \ell^2.$$

Bevisa att $=$ gäller $\iff$ $c_n = \langle f, \phi_n\rangle$ gäller $\forall n \in \mathbb{N}$.

We make a few definitions: let

$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$

and

$$\varphi := \sum c_n\phi_n.$$

Then we compute

$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 + 2\Re\langle f - g, g - \varphi\rangle.$$

I claim that

$$\langle f - g, g - \varphi\rangle = 0.$$

Just write it out:

$$\langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$

$$= \sum \overline{\widehat{f_n}}\langle f, \phi_n\rangle - \sum \overline{c_n}\langle f, \phi_n\rangle - \sum \widehat{f_n}\langle \phi_n, \sum \widehat{f_m}\phi_m\rangle + \sum \widehat{f_n}\langle \phi_n, \sum c_m\phi_m\rangle$$

$$= \sum |\widehat{f_n}|^2 - \sum \overline{c_n}\widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n}\overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have

$$||f - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 \geq ||f - g||^2,$$

with equality iff

$$||g - \varphi||^2 = 0 \iff g = \varphi.$$

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3. Antag att $\{\phi_n\}_{n\in\mathbb{N}}$ är egenfunktionerna med egenvärdena $\{\lambda_n\}_{n\in\mathbb{N}}$ till ett regulärt Sturm-Liouvilleproblem på intervallet $[a,b]$,

$$Lu + \lambda u = 0.$$

Med hjälp av $\{\phi_n\}_{n\in\mathbb{N}}$ och $\{\lambda_n\}_{n\in\mathbb{N}}$, bestämma alla lösningar $u \in \mathcal{L}^2([a,b])$ till:

$$u + Lu = 0, \quad x \in [a,b].$$

Either you know your SLPs or you don't. If you do, then you know that $\{\phi_n\}$ are an ONB for the Hilbert space, $\mathcal{L}^2([a,b])$. Consequently, any u in that Hilbert space which solves the equation is a linear combination of the ϕ_n. Write the equation:

$$u = \sum \widehat{u_n}\phi_n \implies Lu = \sum -\lambda_n \widehat{u_n}\phi_n = -u = \sum -\widehat{u_n}\phi_n$$

$$\iff -\widehat{u_n} = -\lambda_n \widehat{u_n} \quad \forall n \in \mathbb{N}.$$

Now, okay, the trivial solution $u \equiv 0$ is a solution, but it doesn't really count, does it? Otherwise, the only way the above equation holds is that

$$\forall \widehat{u_n} \neq 0 \implies \lambda_n = 1.$$

Consequently, there are non-trivial solutions to this problem $\iff$ 1 is an eigenvalue, in which case all the $\mathcal{L}^2$ solutions are all linear combinations of eigenfunctions ϕ_n with eigenvalue $\lambda_n = 1$.

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4. Beräkna

$$\int_{\mathbb{R}} \frac{t^2}{(t^2+9)(t^2+16)} dt.$$

To begin with, let's call this integral $\star$. If you prefer, you could of course use the residue theorem. Which contour will work here? It seems like the Rainbow ought to do the trick. Otherwise, we can have some fun with the Fourier transform and Plancharel's theorem. Let

$$F(t) = \frac{t}{t^2+3^2}, \quad f(t) = \frac{1}{t^2+3^2}, \quad G(t) = \frac{t}{t^2+4^2}, \quad g(t) = \frac{1}{t^2+4^2}.$$

Then, by Plancharel's Theorem we have

$$\star = \frac{1}{2\pi} \int \widehat{F}\overline{\widehat{G}}.$$

Don't forget that complex conjugation!!! In solving this one, I forgot it the first time, and I kept getting a negative answer, which was really annoying, because *obviously* $\star > 0$.

Next, we use two of the handy formulae, which say that

$$\widehat{F} = \widehat{tf(t)} = i\frac{d}{d\xi}\frac{\pi}{3}e^{-3|\xi|} = -i\pi e^{-3|\xi|},$$

and

$$\widehat{G} = -i\pi e^{-4|\xi|}.$$

What is $\bar{i}$? The complex conjugate of i is of course $-i$. This is important, because we now have

$$\star = \frac{1}{2\pi} \int \left(-i\pi e^{-3|\xi|}\right) \left(\overline{-i\pi e^{-4|\xi|}}\right) d\xi$$
$$= \frac{\pi}{2} \int e^{-7|\xi|} d\xi = \pi \int_0^\infty e^{-7x} dx = \frac{\pi}{7}.$$

♡

5. Sök en begränsad lösning till:

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(0,x) = \frac{1}{x^2+1}.$$

This is *Fourier Analysis,* and who came up with the heat equation? *Fourier.* So, of course we would like to solve it with the Fourier transform, and in this case, we can do so, because $x \in \mathbb{R}$. The Fourier transform is what one uses on $\mathbb{R}$, whereas Fourier series are what one uses on a bounded interval. Please remember that (unless you're dealing with some *other* equation, in which case you may need *other* techniques).

So, recalling the fundamental solution of the heat equation, or if you prefer, working it out for the zillionth time, we have

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int e^{-|x-y|^2/(4t)} \frac{1}{y^2+1} dy.$$

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6. Lös problemet:

$$
\begin{aligned}
(1+t)u_t &= u_{xx}, && 0<x<2, \quad t>0,\\
u(0,t) &= 0, && t>0,\\
u(2,t) &= 0, && t>0,\\
u(x,0) &= 2x, && 0<x<2.
\end{aligned}
$$

Well, this is sort of like a heat equation on a bounded interval. We do one of our favorite things: separate variables. Assume

$$u(x,t) = f(t)g(x).$$

Then, we get

$$(1+t)f'(t)g(x) = f(t)g''(x).$$

Divide both sides by fg, so we have

$$(1+t)\frac{f'}{f}(t) = \frac{g''}{g}(x).$$

One side depends only on t, other side depends only on x, which means both sides are constant. (Why?) Which side looks easiest? The x one! I mean, come on, it has less stuff in it. So of course we start by solving that side and use the boundary conditions!

Hence,

$$g''(x) = \lambda g(x), \quad g(0) = g(2) = 0.$$

Oh geez, you must know the solutions to this guy by now. Yes, indeed, this is a regular SLP, and the (not normalized) eigenfunctions are

$$g = g_n(x) = \sin(n\pi x/2).$$

How do we normalize them? Well, everybody knows that

$$\int_0^2 \sin(n\pi x/2)^2 dx = \frac{2}{2} = 1.$$

Thus the normalized eigenfunctions are in fact the same!! (Yes, I did this intentionally to make it less messy.) So all the possible gs are

$$g_n(x) = \sin(n\pi x/2), \quad \lambda_n = -\frac{n^2\pi^2}{4}.$$

Next, we need to deal with the time business. We need to solve

$$(1+t)\frac{f'(t)}{f(t)} = -\frac{n^2\pi^2}{4}.$$

Well, actually both sides depend on n, so we should incorporate this into the f, writing f_n. Note that the derivative of $\log(f(t))$ is

$$\frac{f'(t)}{f(t)}.$$

So this equation is

$$\log(f_n(t))' = -\frac{n^2\pi^2}{4+4t} \implies \log(f_n(t)) = -\frac{n^2\pi^2}{4}\log(1+t) = \log((1+t)^{-n^2\pi^2/4}),$$

$$\implies f_n(t) = a_n(1+t)^{-n^2\pi^2/4}, \quad a_n \in \mathbb{R}.$$

Thus, we have the following pairs of solutions

$$f_n(t)g_n(x) = a_n(1+t)^{-n^2\pi^2/4}\sin(n\pi x/2).$$

Which piece of information have we not yet used? The $t = 0$ condition! When $t = 0$, $f_n(0) = a_n$. Somehow we need to use the g_n then to get the initial data. This is where Fourier series comes into play. We need

$$a_n = \int_0^2 2x\sin(n\pi x/2)dx = -2x\frac{\cos(n\pi x/2)}{(n\pi/2)}\Big|_{x=0}^2 + \int_0^2 2\frac{\cos(n\pi x/2)}{(n\pi/2)}dx$$

$$= \frac{8(-1)^{n+1}}{n\pi}.$$

Finally, our solution

$$u(x,t) = \sum_{n\geq 1}\frac{8(-1)^{n+1}}{n\pi}(1+t)^{-n^2\pi^2/4}\sin(n\pi x/2).$$

♡

7. Hitta polynomet $p(x)$ av högst grad 1 som minimerar:

$$\int_0^1 |e^x - p(x)|^2 dx.$$

We need to find polynomials of degree 0 and 1, which are orthogonal and have $\mathcal{L}^2$ norm 1 on $[0,1]$. The first one is pretty easy, it is $p_0(x) \equiv 1$. The next one needs to be orthogonal to p_0, so this means since it is degree one it looks like $p_1(x) = ax + b$,

$$\int_0^1 (ax+b)dx = 0 \implies b = -\frac{a}{2}.$$

That is because $\int_0^1 axdx = a/2$ whereas $\int_0^1 bdx = b$. To have norm one requires

$$\int_0^1 (ax - a/2)^2 dx = 1 \implies \left. \frac{(ax-a/2)^3}{3a} \right|_{x=0}^{x=1} = 1$$

$$\implies \frac{(a/2)^3}{3a} - \frac{(-a/2)^3}{3a} = 1 \implies a = 2\sqrt{3}.$$

Thus, similar to problem 2, we compute the coefficients of these polynomials,

$$c_0 = \int_0^1 e^x p_0(x) dx = \int_0^1 e^x dx = e - 1.$$

For the sake of simplicity, we begin next by computing (using integration by parts)

$$\int_0^1 xe^x dx = e - \int_0^1 e^x dx = e - (e-1) = 1.$$

That's rather lovely isn't it?

So, we then know that

$$c_1 = \int_0^1 p_1(x) e^x dx = \int_0^1 (2\sqrt{3}x - \sqrt{3}) e^x dx = 2\sqrt{3} - \sqrt{3}(e-1) = \sqrt{3}(3-e).$$

Hence, the polynomial we seek is

$$\begin{aligned} p(x) &= c_0 p_0(x) + c_1 p_1(x) = (e-1) + \sqrt{3}(3-e)(2\sqrt{3}x - \sqrt{3}) \\ &= 6x(3-e) + e - 1 - 3(3-e) = 6x(3-e) + 4e - 10. \end{aligned}$$

Who would possibly guess that? It doesn't seem particularly obvious, does it?

8. Låt

$$I_0(x) = \sum_{m\geq 0} \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}.$$

Visa att gäller:

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos(\theta)}d\theta.$$

Well, you know since kindergarten that

$$e^x = \sum_{n\geq 0}\frac{x^n}{n!},$$

so of course

$$\int_0^\pi e^{x\cos(\theta)}d\theta = \int_0^\pi \sum_{n\geq 0}\frac{(x\cos(\theta))^n}{n!}d\theta = \sum_{n\geq 0}\frac{x^n}{n!}\int_0^\pi \cos(\theta)^n d\theta.$$

This re-arranging is no problem because this series converges absolutely and uniformly on compact subsets of $\mathbb{C}$. You can probably guess from the formula page that you should write

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

and use the binomial theorem

$$\cos(\theta)^n = 2^{-n}\sum_{k=0}^n \binom{n}{k} e^{ik\theta}e^{i(n-k)(-\theta)} = 2^{-n}\sum_{k=0}^n \binom{n}{k} e^{i\theta(2k-n)}.$$

Well, it's easy enough to integrate

$$\int_0^\pi e^{i\theta(2k-n)}d\theta = \pi, \quad k = \frac{n}{2}, \quad \frac{e^{i\pi(2k-n)}-1}{2k-n} \text{ otherwise.}$$

Since k is an integer, $k = n/2$ is possible iff n is even.

Since

$$e^{2\pi ik} = 1,$$

$$\frac{e^{i\pi(2k-n)}-1}{2k-n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-2}{2k-n} & \text{if } n \text{ is odd}\end{cases}$$

Thus,

$$\int_0^\pi \cos(\theta)^n d\theta = 2^{-n}\sum_{k=0}^n \binom{n}{k}\int_0^\pi e^{i\theta(2k-n)}d\theta = 2^{-n}\binom{n}{n/2}\pi \text{ for } n \text{ even,}$$

and

$$2^{-n}\left(\sum_{k<n/2}\binom{n}{k}\frac{-2}{2k-n}+\sum_{k>n/2}\binom{n}{k}\frac{-2}{2k-n}\right), \qquad \text{for } n \text{ odd.}$$

By symmetry of the binomial coefficients, the two sums above cancel perfectly, creating a lovely and simple 0. Thus,

$$\int_0^\pi \cos(\theta)^n d\theta = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^{-n}\binom{n}{n/2}\pi & \text{if } n \text{ is even} \end{cases}$$

Hence,

$$\int_0^\pi e^{x\cos(\theta)} d\theta = \sum_{n\geq 0}\frac{x^n}{n!}c_n, \quad c_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^{-n}\binom{n}{n/2}\pi & \text{if } n \text{ is even} \end{cases}$$

Dividing by π and making the substitution $n = 2m$,

$$\frac{1}{\pi}\int_0^\pi e^{x\cos(\theta)} d\theta = \sum_{m\geq 0}\frac{\left(\frac{x}{2}\right)^{2m}}{(2m)!}\binom{2m}{m}.$$

By definition,

$$\binom{2m}{m} = \frac{(2m)!}{m!m!}.$$

So, this simplifies to

$$\frac{1}{\pi}\int_0^\pi e^{x\cos(\theta)} d\theta = \sum_{m\geq 0}\frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2} = I_0(x).$$

This might seem like a contrite fact, but such manipulations could save the planet one day.

♡

Formler:

1. $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$
2. $\widehat{fg}(\xi) = (2\pi)^{-1}(\hat{f} * \hat{g})(\xi)$
3. $\widehat{e^{-ax^2/2}}(\xi) = \sqrt{\frac{2\pi}{a}} e^{-\xi^2/(2a)}$
4. $\widehat{xf(x)}(\xi) = i\frac{d}{d\xi}\widehat{f}(\xi)$
5. En föjld $\{c_n\}_{n\in\mathbb{N}}$ med $c_n \in \mathbb{C}$ $\forall n \in \mathbb{N}$ är i ℓ^2 $\iff$
$$\sum_{n\in\mathbb{N}} |c_n|^2 < \infty.$$
6. $\widehat{\frac{1}{x^2+a^2}}(\xi) = (\pi/a)e^{-a|\xi|}$.
7. $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$.
8. Binomial sats: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$, med $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.