Fourieranalys MVE030 och Fourier Metoder MVE290 lp3 5.april.2016
Betygsgränser: 3: 40 poäng, 4: 50 poäng, 5: 60 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA och en typgodkänd räknedosa.
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1. Bevisa:

$$e^{xz/2-x/(2z)} = \sum_{n\in\mathbb{Z}} J_n(x)z^n, \quad J_n \text{ är Besselfunktionen av grad } n.$$

   This is the proof of the generating function for the Bessel functions. See Folland §5.2 p. 135 which contains the proof in all its glory.

   (10 p)

2. Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortogonala i ett Hilbert-rum, H. Bevisa att följande tre är ekvivalenta:

$$(1) \quad f \in H \text{ och } \langle f, \phi_n\rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

   (10 p)

   This is essentially Theorem 3.4 in Folland (that Theorem is for the Hilbert space $L^2(a,b)$, but the proof is identical for a general Hilbert space); see the proof on p. 77 in §3.3 of Folland.

3. Antag att $\{\phi_n\}_{n\in\mathbb{N}}$ är egenfunktionerna med egenvärdena $\{\lambda_n\}_{n\in\mathbb{N}}$ till ett regulärt Sturm-Liouvilleproblem på intervallet $[a,b]$,

$$Lu + \lambda u = 0.$$

   Med hjälp av $\{\phi_n\}_{n\in\mathbb{N}}$ och $\{\lambda_n\}_{n\in\mathbb{N}}$, lös:

$$u_t + Lu = 0, \quad t \geq 0, \, x \in [a,b],$$

$$u(0,x) = f(x) \in L^2([a,b]), \quad \int_a^b |f(x)|^2 dx < \infty.$$

(10 p)

This is a heat equation problem. It is just like doing separation of variables on the interval $[a,b]$ to solve

$$u_t + u_{xx} = 0, \quad u(0,x) = f(x).$$

If one does separation of variables in our problem,

$$T'(t)X(x) + T(t)L(X)(x) = 0 \iff \frac{T'}{T} = -\frac{L(X)}{X} = \lambda \iff L(X) + \lambda X = 0.$$

The solutions to this are provided above, they are

$$X = \phi_n, \quad \lambda = \lambda_n.$$

Therefore the corresponding $T(t)$ satisfies

$$T_n' = \lambda_n T_n \implies T_n(t) = a_n e^{\lambda_n t}.$$

The solution is given by

$$u(t,x) = \sum_{n \geq 1} e^{\lambda_n t} \phi_n(x) a_n,$$

with

$$a_n = \int_a^b f(x)\phi_n(x)dx.$$

4. Lös:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(0,y) = u(1,y) = 0,$$
$$u(x,0) = f(x), \quad u(x,1) = g(x),$$

med f och g kontuerliga på $[0,1]$. (10 p)

Separation of variables, since it's a bounded interval, this is going to use Sturm-Liouville methods. So, we have

$$X''Y + Y''X = 0, \quad X(0) = X(1) = 0.$$

Thus, we solve for X first, since it's easiest, and we get the solutions

$$X_n(x) = \sin(n\pi x), \quad n \in \mathbb{N}.$$

This gives us the constant, so we can solve for Y,

$$\frac{Y''}{Y} + \frac{X''}{X} = 0 \iff \frac{Y''}{Y} - n^2\pi^2 = 0 \iff Y'' = n^2\pi^2 Y.$$

Hence the corresponding

$$Y_n = a_n \cosh(n\pi y) + b_n \sinh(n\pi y).$$

To solve for the constants, we use the boundary conditions. We let

$$u(x,y) = \sum_{n\geq 1} \sin(n\pi x)(a_n \cosh(n\pi y) + b_n \sinh(n\pi y)).$$

At $y = 0$ we need

$$u(x,0) = \sum_{n\geq 1} a_n \sin(n\pi x) = f(x).$$

Thus, we want

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)dx.$$

This is because the L^2 norm of $\sin(n\pi x)$ on an interval of length 1 is 1/2. Next, we want

$$u(x,1) = \sum_{n\geq 1} \sin(n\pi x)(a_n \cosh(n\pi) + b_n \sinh(n\pi)) = g(x).$$

Thus, we let

$$c_n = 2\int_0^1 g(x)\sin(n\pi x)dx = a_n \cosh(n\pi) + b_n \sinh(n\pi),$$

and so therefore

$$b_n = \frac{2\int_0^1 g(x)\sin(n\pi x)dx - 2\int_0^1 f(x)\sin(n\pi x)dx \cosh(n\pi)}{\sinh(n\pi)}.$$

5. Sök en begränsad lösning till:

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(0,x) = x^2 e^{-x^2}.$$

(10 p)

What a surprise, the heat equation. Since it is on $\mathbb{R}$, and after all Fourier derived the heat equation, it makes sense to use the Fourier transform. We do that to the equation and it becomes

$$\widehat{u_t(t,\xi)} = -\xi^2 \hat{u}(t,\xi).$$

This is just an ODE for t, and so we solve it to find

$$\hat{u}(t,\xi) = a(\xi)e^{-\xi^2 t}.$$

By the initial conditions, we know that

$$a(\xi) = \widehat{x^2 e^{-x^2}}(\xi).$$

Thus

$$\hat{u}(t,\xi) = \widehat{x^2 e^{-x^2}}(\xi)e^{-\xi^2 t}.$$

We know that the Fourier transform of a convolution is a product. Hence, if we can find a function g whose Fourier transform is $e^{-\xi^2 t}$, then we know that

$$u(t,x) = \int_{\mathbb{R}} g(x-y)y^2 e^{-y^2} dy.$$

Well, this follows from Formula number 3, so we have

$$u(t,x) = \int_{\mathbb{R}} (4\pi t)^{-1/2} e^{-(x-y)^2/(4t)} y^2 e^{-y^2} dy.$$

Challenge: Solve this!! (not required however)

6. Antag att

$$\int_{-4}^{5} |f(x)|^2 dx < \infty.$$

Bestäm:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos(nx)dx.$$

Motivera ditt svar! (10 p)

Well, well, well, what have we here? We have an element $f \in \mathcal{L}^2(-\pi,\pi)$, since

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \le \int_{-4}^{5} |f(x)|^2 dx < \infty.$$

What is an orthonormal basis for $L^2(-\pi,\pi)$? That's right

$$\phi_n = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{Z}.$$

Thus, we know by PROBLEM ONE on this very exam that

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = ||f||^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \implies \lim_{n\to\pm\infty} |c_n|^2 = 0 \implies \lim_{n\to\pm\infty} c_n = 0$$

$$\implies \lim_{n\to\pm\infty} \Re c_n = 0 \text{ and } \lim_{n\to\pm\infty} \Im c_n = 0 \implies \lim_{n\to\infty} \sqrt{2\pi}\Re c_n = 0.$$

Above,

$$c_n = \int_{-\pi}^{\pi} f(x)\overline{\phi_n(x)} dx = \langle f, \phi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi} \pi f(x) e^{-inx} dx,$$

and so we have

$$\int_{-\pi}^{\pi} f(x)\cos(nx) dx = \sqrt{2\pi}\Re c_n \to 0 \text{ as } n \to \pm\infty.$$

We have just *proven* that the limit we seek is zero.

7. Hitta polynomet $p(x)$ av högst grad 2 som minimerar

$$\int_{-2}^{2} |x^5 - p(x)|^2 dx.$$

(10 p)

Best approximation problem, plain and simple. The polynomial x^5 is odd, so we know that

$$\langle x^5, p(x) \rangle = 0$$

for any polynomial $p(x)$ which is *even.* We start by computing the first polynomial in our L^2 ONB of polynomials:

$$\int_{-2}^{2} 1 dx = 4 \implies p_0 = \frac{1}{2}.$$

Then, $p_1(x) = ax + b$. Orthogonality to p_0 requires that

$$\langle p_1, p_0 \rangle = 0 \iff \int_{-2}^{2} (ax+b) dx = 0 \iff b = 0.$$

To solve for a we then use that we wish for L^2 norm equal to one, thus we want

$$\int_{-2}^{2} a^2x^2dx = 1 \iff a^2\frac{2^4}{3} = 1 \iff a = \frac{\sqrt{3}}{4} \implies p_1(x) = \frac{\sqrt{3}}{4}x.$$

The polynomial $p_2(x) = ax^2 + bx + c$. However, we can save ourselves a bit of work, by noting that orthogonality to p_1 requires

$$\langle x, p_2\rangle = 0 \iff b = 0.$$

Hence, we know that

$$\langle x^5, p_j\rangle = 0, \quad j = 0, 2.$$

We compute that

$$\langle x^5, p_1\rangle = 2\int_0^2 \frac{\sqrt{3}}{4}x^6dx = \frac{\sqrt{3}2^6}{7},$$

thus the polynomial we seek is

$$\frac{\sqrt{3}2^6}{7}p_1(x) = \frac{3(2^4)}{7}x.$$

8. Låt $f(x) = e^x$, $-\pi < x \leq \pi$, och förläng f till en 2π-periodisk funktion på $\mathbb{R}$.
   (a) Bestäm Fourier-serien av f.
   (b) Gäller det att Fourier-koefficienterna till f', c_n', upfyller $c_n' = inc_n$, där c_n är Fourier-koefficienterna till f? Motivera ditt svar!

(10 p)

We simply compute

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{2\pi(1-in)}$$

and for $|x| < \pi$, we have

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} = \sum_{n\in\mathbb{Z}} \left(\frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{2\pi(1-in)}\right) e^{inx}.$$

Differentiating termwise is *rubbish!* The Fourier coefficients of f *are the same* as the Fourier coefficients of f', because the derivative of e^x is just e^x again. Since $f \in L^2(-\pi, \pi)$, therefore the expansion of f in terms of the ONB

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}}$$

is *unique.* This expansion is precisely

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} = \sum_{n\in\mathbb{Z}} \left(\frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{2\pi(1-in)} \right) e^{inx}.$$

Since $f'(x) = f(x)$, by the uniqueness of the expansion, $c_n' = c_n \neq inc_n$.

Formler:

1. $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$
2. $\widehat{fg}(\xi) = (2\pi)^{-1}(\hat{f} * \hat{g})(\xi)$
3. $\widehat{e^{-ax^2/2}}(\xi) = \sqrt{\frac{2\pi}{a}} e^{-\xi^2/(2a)}$
4. Bessel funktionen

$$J_n(x) = \sum_{k\geq 0} (-1)^k \frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\Gamma(k+n+1)}$$