Fourieranalys/Fourier Metoder lp3 18:e mars, 2016
Betygsgränser: 3: 40 poäng, 4: 50 poäng, 5: 60 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA och en typgodkänd räknedosa.
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1. Bevisa Samplingsatsen: Låt $f \in \mathcal{L}^2(\mathbb{R})$ och låt $\hat{f}$ vara Fouriertransformen av f. Antag att det finns $L > 0$ sådant $\hat{f}(x) = 0 \ \forall x \in \mathbb{R}$ med $|x| > L$. Visa att

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

(10 p)

Proof. This theorem is all about the interaction between Fourier series and Fourier coefficients and how to work with both simultaneously. Since the Fourier transform $\hat{f}$ has compact support, the following equality holds as elements of $\mathcal{L}^2([-L, L])$,

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx.$$

We shall next use the Fourier inversion theorem (FIT) to write

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixt} \hat{f}(x) dx = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \hat{f}(x) dx.$$

On the left we have used the fact that $\hat{f}$ is supported in the interval $[-L, L]$, thus the integrand is zero outside of this interval, so we can throw that part of the integral away.

Now, we substitute the Fourier expansion of $\hat{f}$ into this integral,

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} c_n e^{in\pi x/L} dx.$$

Let us take a closer look at the coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx = \frac{1}{2L} \int_{\mathbb{R}} e^{ix(-n\pi/L)} \hat{f}(x) dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

In the second equality we have used the fact that $\hat{f}(x) = 0$ for $|x| > L$, so by including that part we don't change the integral. In the third equality we have used the FIT!!! So, we now substitute this into our formula above for

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L} dx$$

This is approaching the form we wish to have in the theorem, but the argument of the function f has a pesky negative sign. That can be remedied by switching the order of summation, which does not change the sum, so

$$f(t) = \frac{1}{2L} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{-in\pi x/L} dx.$$

We may also interchange the summation with the integral[1]

$$f(t) = \frac{1}{2L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \int_{-L}^{L} e^{x(it - in\pi/L)} dx.$$

We then compute

$$\int_{-L}^{L} e^{x(it-in\pi/L)} dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)} \sin(Lt - n\pi).$$

Substituting,

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \frac{\sin(Lt - n\pi)}{Lt - n\pi}.$$

$\square$

2. Låt f vara en 2π-periodisk funktion med $f \in C^2(\mathbb{R})$. Bevisa att Fourierkoefficienterna c_n av f och Fourierkoefficienterna c_n' av f' uppfyller

$$c_n' = inc_n.$$

(10 p)

[1] None of this makes sense pointwise; we are working over $\mathcal{L}^2$. The key property which allows interchange of limits, integrals, sums, derivatives, etc is *absolute convergence.* This is the case here because elements of $\mathcal{L}^2$ have $\int |f|^2 < \infty$. That is precisely the type of absolute convergence required.

We quite simply use the definitions of the Fourier series and coefficients of f and f' respectively. By the hypothesis,

$$f(x) = \sum_{\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

and

$$f'(x) = \sum_{\mathbb{Z}} c_n' e^{inx}, \quad c_n' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx.$$

Integrating by parts and using the periodicity of f and consequently also f' as well as the periodicity of e^{-inx} we have

$$c_n' = \frac{1}{2\pi} \int_{-\pi}^{\pi} -f(x)(-in) e^{-inx} dx = inc_n.$$

3. Lös

$$u_t = u_{xx}, \quad t > 0,\ x \in (0, \pi),$$
$$u(0, x) = \pi x - x^2, \quad u(t, 0) = u(t, \pi) = 0.$$

(10 p)

Notice how the fact that $x \in (0, \pi)$ just SCREAMS at you to use Fourier series to solve this. Either you can see directly that it's gonna be sines, due to the Dirichlet Boundary Conditions $u(t, 0) = u(t, \pi) = 0$, or you do it by hand the old fashioned way starting off with separation of variables. If you do that, you assume

$$u(t, x) = T(t)X(x).$$

The heat equation becomes

$$T'X = TX'' \iff \frac{T'}{T} = \frac{X''}{X}. \tag{1}$$

Ergo both sides must be constant. Which side to solve first? The one with the most easy information. We have

$$X(0) = X(\pi) = 0.$$

The only solutions to X'' = constant times X which satisfy these conditions are

$$X = X_n = \sin(nx),$$

as well as constant multiples of these. We shall deal with the constants later.

Now, however, we can use this to find the partner $T = T_n$ who is paired up with X_n, due to the equation (1). From the X_n side, we get that $T'/T = X''/X = -n^2$. So, we have the equation

$$T_n' = -n^2 T_n \implies T_n(t) = a_n e^{-n^2 t}.$$

Above, a_n is a constant. Now, we take our solution

$$u(t,x) = \sum_{n \geq 1} a_n e^{-n^2 t} \sin(nx). \tag{2}$$

To determine the a_n we use the IC (initial condition),

$$u(0,x) = \pi x - x^2 = \sum_{n \geq 1} a_n \sin(nx).$$

So, we need to expand the function $\pi x - x^2$ as a sine series. The L^2 norm of $\sin(nx)$ on an interval is as usual half the length of that interval, so in this case it is $\frac{\pi}{2}$. We then compute for the sake of simplicity first

$$\begin{aligned} b_n = \int_0^\pi x \sin(nx) dx &= -x \frac{\cos(nx)}{n} |_0^\pi + \int_0^\pi \frac{\cos(nx)}{n} dx \\ &= -\frac{\pi(-1)^n}{n}. \end{aligned}$$

We have used integration by parts.

Next we use integration by parts twice to compute

$$\begin{aligned} c_n = \int_0^\pi x^2 \sin(nx) dx &= -x^2 \frac{\cos(nx)}{n} |_0^\pi + \int_0^\pi 2x \frac{\cos(nx)}{n} dx \\ &= -\frac{\pi^2(-1)^n}{n} + 2x \frac{\sin(nx)}{n^2} |_0^\pi - \int_0^\pi 2 \frac{\sin(nx)}{n^2} dx \\ &= -\frac{\pi^2(-1)^n}{n} + 2 \frac{\cos(nx)}{n^3} |_0^\pi = -\frac{\pi^2}{n} \text{ if } n \text{ is even} \end{aligned}$$

or

$$\frac{\pi^2}{n} - \frac{4}{n^3} \text{ if } n \text{ is odd.}$$

We thus have

$$a_n = \frac{2}{\pi} (\pi b_n - c_n) = \frac{8}{\pi n^3} \ n \text{ odd, even terms are all zero.}$$

One can then insert this into (2) to obtain the solution.

4. Lös
$$u_t = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}.$$
$$u(0,x) = \frac{1}{1+x^2}.$$
(10 p)

Doesn't this just SCREAM Fourier transform? If we take the Fourier transform of the heat equation, we get
$$\partial_t \hat{u}(t,\xi) = -\xi^2 \hat{u}(t,\xi).$$
This is just an ODE in the variable t. So, we solve it to get
$$\hat{u}(t,\xi) = a(\xi)e^{-\xi^2 t},$$
where $a(\xi)$ is a function that depends only on ξ and not on t. Then, we use the initial condition
$$\hat{u}(0,\xi) = \widehat{\frac{1}{1+x^2}}(\xi) \implies a(\xi) = \widehat{\frac{1}{1+x^2}}(\xi).$$
Now, we think about the Formula # 1. The Fourier transform of a convolution is the product of the Fourier transforms. Using Formula # 3, we see that the Fourier transform of
$$g(t,x) = (4\pi t)^{-1/2} e^{-x^2/4t}$$
is $e^{-\xi^2 t}$. Thus we have
$$g * \widehat{f(t,x)}(\xi) = a(\xi)e^{-\xi^2 t},$$
and therefore
$$u(t,x) = g * f(t,x) = \int_{\mathbb{R}} \frac{1}{1+y^2} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy.$$
Challenge! This is *not* required to receive full points on the exam, but it is a little extra for the students who get bored easily. Can you compute this convolution?

5. Lös
$$u_{tt} = u_{xx}, \quad x > 0, \quad t > 0,$$
$$u(0,x) = 0, \quad u_t(0,x) = 0,$$

$$u(t,0) = (1+t)^{3/2}.$$

(10 p)

Well, well, what do we have here? Half lines? Conditions like $u(0,x) = u_t(0,x) = 0$? What does that scream at us? Yes, Laplace transform! Let us Laplace transform the wave equation here

$$z^2\tilde{u}(z,x) = \partial_x^2 \tilde{u}(z,x).$$

This is just an ODE for $\tilde{u}$ with respect to the variable x. We know that a basis of solutions are

$$\tilde{u}(z,x) = e^{\pm zx}a(z),$$

where $a(z)$ depends only on z, not on x. We just need to solve, i.e. find *a* solution. So, let's choose one of these, and I like e^{-zx}, because it is physically reasonable. Then, we have

$$\tilde{u}(z,x) = e^{-zx}a(z).$$

Using the initial condition

$$\tilde{u}(0,x)(z) = a(z) = \widetilde{(1+t)^{3/2}}(z).$$

If we can therefore find a function whose Laplace transform is

$$e^{-zx}\widetilde{(1+t)^{3/2}}(z),$$

then that function is a solution! Looking at Formula #4 is rather helpful. It shows us that $H(t-x)f(t-x)$ has the desired Laplace transform, where

$$f(t) = (1+t)^{3/2},$$

and H is the Heavyside function. Thus our solution is

$$u(t,x) = (1+(t-x))^{3/2}, \quad t > x, \quad 0t \le x.$$

6. Legendrepolynomen

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \ge 0,$$

är en ortogonal bas på $L^2([-1,1])$ med

$$||P_n||_{L^2}^2 = \frac{2}{2n+1}.$$

Antag att f är kontinuerlig på $(-2, 2)$, beräkna

$$\lim_{n\to\infty} \sqrt{\frac{2n+1}{2}} \int_{-1}^{1} f(x)P_n(x)dx.$$

(10 p)

This is actually a theory problem masquerading as a special functions problem. So, we know that these P_n are an orthogonal basis for L^2 on the interval $[-1, 1]$. By the assumption that f is continuous on the *larger* interval $(-2, 2)$, f is uniformly continuous on the closed interval $[-1, 1]$ and it is also bounded there. Hence, it's in L^2. Hence, we can write it using a basis for L^2. We see that the P_n do not have L^2 norm equal to one, so let us define

$$\phi_n = \frac{P_n}{||P_n||} = \sqrt{\frac{2n+1}{2}} P_n.$$

Then, the set $\{\phi_n\}_{n\geq 0}$ is an orthonormal basis for L^2 of $[-1, 1]$. We can express f in terms of this basis with

$$c_n = \langle f, \phi_n \rangle = \sqrt{\frac{2n+1}{2}} \int_{-1}^{1} f(x)P_n(x)dx.$$

Note that the complex conjugate in the definition of the inner product is not there because everything is real. For real. Now, you can either use Bessel's inequality, or the fact that since it's an ONB, we have

$$\infty > ||f||^2 = \int_{-1}^{1} |f(x)|^2 dx = \sum_{n\geq 0} c_n^2.$$

The terms in any convergent sum tend to 0, thus

$$\lim_{n\to\infty} c_n^2 = 0 \implies \lim_{n\to\infty} c_n = 0.$$

So, the limit we seek is 0.

7. Hitta det polynom av högst grad 2 som minimerar

$$\int_{-1}^{1} |\sin(\pi x) - p(x)|^2 dx.$$

(10 p)

One could easily be tempted to use the Legendre polynomials as orthonormal basis polynomials here, and I started doing it that way, and it got all messy. So, I prefer the following solution, which is simpler. We need to find the first three orthonormal polynomials, of degrees 0, 1, and 2, respectively, on $L^2([-1,1])$. The first one is of degree zero thus it is a constant, and since we need it to have L^2 norm 1, we compute

$$\int_{-1}^{1} c^2 dx = 2c^2 \implies p_0 = \frac{1}{\sqrt{2}}.$$

Next, we compute the polynomial of degree one which is orthogonal to p_0 and also has L^2 norm 1. Such a polynomial is of the form $p_1(x) = ax + b$. Orthogonality to p_0 requires

$$\int_{-1}^{1} \frac{1}{\sqrt{2}}(ax+b)dx = 0 \iff b = 0.$$

Next, we wish to have L^2 norm one, and thus we compute

$$\int_{-1}^{1} a^2 x^2 dx = 2a^2 \int_{0}^{1} x^2 dx = \frac{2a^2}{3} \implies a = \sqrt{\frac{3}{2}}.$$

So,

$$p_1(x) = \sqrt{\frac{3}{2}} x.$$

Finally, for the orthogonality condition on the polynomial of p_2, with respect to p_1, we have for a generic degree two polynomial, ax^2+bx+c

$$\int_{-1}^{1} (ax^2+bx+c)\sqrt{\frac{3}{2}} x dx = 0 \iff b = 0.$$

We can spare ourselves some work here. The polynomial we seek is

$$a_0 p_0 + a_1 p_1 + a_2 p_2,$$

where

$$a_k = \int_{-1}^{1} \sin(\pi x) p_k(x) dx = \langle \sin(\pi x), p_k \rangle.$$

Using the fact that the sine is an odd function,

$$a_0 = 0, \quad a_2 = \int_{-1}^{1} \sin(\pi x)(ax^2+c)dx = 0.$$

Thus, we only need to compute

$$
\begin{aligned}
a_1 = \sqrt{\frac{3}{2}} \int_{-1}^{1} x \sin(\pi x) dx &= \sqrt{6} \int_{0}^{1} x \sin(\pi x) dx \\
&= -\sqrt{6} x \frac{\cos(\pi x)}{\pi} \Big|_0^1 + \sqrt{6} \int_0^1 \frac{\cos(\pi x)}{\pi} dx \\
&= \frac{\sqrt{6}}{\pi}.
\end{aligned}
$$

The polynomial we seek is

$$
\frac{\sqrt{6}}{\pi} \sqrt{\frac{3}{2}} x = \frac{3}{\pi} x.
$$

8. Låt H vara halvskivan

$$
H = \{(x, y) \in \mathbb{R}^2 : y \geq 0, \quad x^2 + y^2 \leq 1\}.
$$

Hitta alla $\lambda < 0$ och funktioner $f \not\equiv 0$ sådana att det i polära koordinater (r, θ) gäller att

$$
\begin{cases}
f_{rr} + r^{-1} f_r + r^{-2} f_{\theta\theta} = -\lambda f & \text{på } H\text{, och} \\
f = 0 & \text{på } \partial H.
\end{cases}
$$

(10 p)

This is classical separation of variables using polar coordinates. Write

$$
f(r, \theta) = R(r)\Theta(\theta).
$$

The equation becomes

$$
R''\Theta + r^{-1} R'\Theta + r^{-2} R\Theta'' = -\lambda R\Theta.
$$

Divide both sides by $R\Theta$ and multiply by r^2.

$$
r^2 \frac{R''}{R} + r \frac{R'}{R} + \frac{\Theta''}{\Theta} = -r^2 \lambda.
$$

Now, if we subtract the Θ''/Θ from both sides we get

$$
r^2 \frac{R''}{R} + r \frac{R'}{R} + r^2 \lambda = -\frac{\Theta''}{\Theta} \implies \text{both sides are equal a constant.} \quad (3)
$$

Which part of the equation is *simplest?* (Well, the constant, but still, I am talking about the Θ part!). Always start simple.

The boundary conditions require that

$$\Theta(0) = \Theta(\pi) = 0.$$

If you don't see this, draw a picture of a half disk, centered at the origin, like in the definition of H. The bottom flat boundary corresponds to $\theta = 0$ on the right half and $\theta = \pi$ on the left half. So, we're looking for functions Θ with Θ'' equal to a constant times Θ and $\Theta(0) = \Theta(\pi) = 0$. Look familiar? It's not a coincidence! The only solutions to this are constant multiples of

$$\Theta = \Theta_n(\theta) = \sin(n\theta).$$

Then we have

$$\frac{\Theta_n''}{\Theta_n} = -n^2.$$

So, this tells us what constant to put into the equation (3) to solve for R. We do this, and we are getting

$$r^2 \frac{R''}{R} + r\frac{R'}{R} + r^2\lambda = n^2 \iff r^2 R'' + rR' + (r^2\lambda - n^2)R = 0.$$

This is so close to Bessel's equation, but not quite. Since we are told to look for solutions with $\lambda > 0$, we can write

$$\lambda = \mu^2, \quad \mu > 0.$$

Let

$$R(r) = f(\mu r), \quad R'(r) = \mu f'(\mu r), \quad R''(r) = \mu^2 f''(\mu r), \quad x = \mu r.$$

Then, our equation for R becomes

$$x^2 f''(x) + xf'(x) + (x^2 - n^2)f(x) = 0. \tag{4}$$

Let us check out the very last Formula. It tells us that this is Bessel's equation of order n. The Bessel function J_n solves it. So, we have a solution given by

$$f(x) = J_n(x) \implies R = R_n(r) = J_n(\mu r) \text{ solves (4).}$$

What remains is to determine μ and therefore $\lambda = \mu^2$. To obtain this, we use the boundary condition. We need to use R_n to make the solution vanish along the circular arc, which is at $r = 1$. Therefore, we need

$$R_n(1) = J_n(\mu) = 0.$$

How do we do this? The Bessel functions have loads of positive zeros, similar to how sines and cosines have loads of positive zeros. So, let $\mu = \mu_{n,k}$ be the k^{th} positive zero of J_n. Then this guarantees that

$$J_n(\mu_{n,k}) = 0.$$

Consequently, we actually have lots of $R_{n,k} = J_n(\mu_{n,k} r)$ for each of the $\Theta_n(\theta) = \sin(n\theta)$. Our set of solutions to the problem are

$$u_{n,k}(r,\theta) = J_n(\mu_{n,k} r)\sin(n\theta), \quad \lambda_{n,k} = \mu_{n,k}^2, \quad n, k \in \mathbb{N}.$$

Formler:

1. $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$
2. $\widehat{fg}(\xi) = (2\pi)^{-1}(\hat{f} * \hat{g})(\xi)$
3. $\widehat{e^{-ax^2/2}}(\xi) = \sqrt{\frac{2\pi}{a}} e^{-\xi^2/(2a)}$
4. $\mathcal{L}\left(H(t-a)f(t-a)(z)\right) = e^{-az}\mathcal{L}(f(z))$, där H är Heavysidefunktionen.
5. Bessels ekvation av ordning n: $x^2 f'' + x f' + (x^2 - n^2) f = 0$.