SOLUTIONS! Fourieranalys MVE030 och Fourier Metoder MVE290 7.juni.2017

Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara en ortonormal mängd i ett Hilbert-rum, H. Om $f \in H$, bevisa att

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

och $=$ gäller $\iff$ $c_n = \langle f, \phi_n\rangle$ gäller $\forall n \in \mathbb{N}$.

Finns i bevis samlingen.

(10 p)

2. Bevisa att Hermite polynomen, $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$, uppfyller $\forall x \in \mathbb{R}$ och $z \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Finns i bevis samlingen.

(10 p)

3. Beräkna:

$$\sum_{n=2}^{\infty} \frac{1}{1+n^2}.$$

(Hint: Utveckla e^x i Fourier-series i intervallet $(-\pi, \pi)$).

Finns i lösningar till tentan 17:e mars, 2017, uppgift 3. Man måste bara ta ut $n = 1$ termen. Altså summan blir

$$\frac{\pi \cosh(\pi)}{2\sinh(\pi)} - \frac{1}{2} - \frac{1}{2} = \frac{\pi \cosh(\pi)}{2\sinh(\pi)} - 1.$$

(10 p)

4. Hitta siffrorna a_0, a_1, och $a_2 \in \mathbb{C}$ som minimerar

$$\int_0^{\pi} |e^x - a_0 - a_1\cos(x) - a_2\cos(2x)|^2 dx.$$

(10 p)

So, we're finding the Fourier-cosine coefficients of e^x basically. Alternatively, we know that the functions $\{\cos(nx)\}_{n\in\mathbb{N}}$ (here I mean Swedish $\mathbb{N}$:-) are an orthogonal basis for $L^2[0,\pi]$. They are not, however, normalized. The L^2 norm is

$$\int_0^\pi 1dx = \pi, \quad \int_0^\pi \cos(nx)^2 dx = \int_0^{n\pi} \cos(\theta)^2 \frac{d\theta}{n}$$

$$= \frac{1}{n}\int_0^{n\pi} \left(\frac{\cos(2\theta)+1}{2}\right) d\theta = \frac{1}{n}\left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right)\Big|_{\theta=0}^{n\pi} = \frac{\pi}{2}.$$

Thus the first three elements in our L^2 ONB are:

$$\frac{1}{\sqrt{\pi}}, \quad \frac{\cos(x)\sqrt{2}}{\sqrt{\pi}}, \quad \frac{\cos(2x)\sqrt{2}}{\sqrt{\pi}}.$$

We next compute the first three Fourier coefficients of e^x with respect to this L^2 ONB,

$$c_0 = \frac{1}{\sqrt{\pi}}\int_0^\pi e^x dx = \frac{e^\pi - 1}{\sqrt{\pi}}.$$

$$c_1 = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\pi e^x \cos(x) dx,$$

$$c_2 = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\pi e^x \cos(2x) dx,$$

so we now compute for $k \in \mathbb{N}$,

$$\int_0^\pi e^x \cos(kx) dx = \Re \int_0^\pi e^x e^{ikx} dx = \Re \frac{e^{x(1+ki)}}{1+ki}\Big|_0^\pi$$

$$= \Re(1-ki)\frac{e^{\pi(1+ik)}-1}{1+k^2} = \Re(1-ki)\frac{e^\pi(-1)^k - 1}{1+k^2} = \frac{(-1)^k e^\pi - 1}{1+k^2}.$$

Setting $k=1$ and $k=2$ we have

$$c_1 = \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{-e^\pi - 1}{2}\right), \quad c_2 = \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{e^\pi - 1}{5}\right).$$

The best approximation is

$$c_0\frac{1}{\sqrt{\pi}} + c_1\frac{\sqrt{2}\cos(x)}{\sqrt{\pi}} + c_2\frac{\sqrt{2}\cos(2x)}{\sqrt{\pi}}$$

which shows that

$$a_0 = c_0 \frac{1}{\sqrt{\pi}}, \quad a_k = c_k \frac{\sqrt{2}}{\sqrt{\pi}}, \quad k = 1, 2.$$

Just because it is rather satisfying, let us write these out using our calculation of the cs above,

$$a_0 = \frac{e^\pi - 1}{\pi}, \quad a_1 = -\frac{(e^\pi + 1)}{\pi}, \quad a_2 = \frac{2(e^\pi - 1)}{5\pi}.$$

5. Lös problemet:

$$u_t - u_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x, 0) = e^{-|x|}$$

(10 p)

Lovely. Initial value problem for the heat equation. The solution is given by the convolution of the initial data with the heat kernel, thus

$$u(t, x) = (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-|y|} e^{-(x-y)^2/4t} dy.$$

You're welcome.

6. Låt $\alpha > 0$. Vi definerar

$$\widehat{LP_\alpha}(f)(\xi) := \hat{f}(\xi) \chi_{(-\alpha,\alpha)}(\xi).$$

Vi definerar

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad \chi_{(-\alpha,\alpha)}(\xi) = \begin{cases} 1 & |\xi| < \alpha \\ 0 & |\xi| \geq \alpha \end{cases}$$

Beräkna $LP_\alpha(f)$ med

$$f(x) = e^{-|x|}.$$

(10 p) I don't know about you, but even though this is pretty simple, it still kinda confuses me with all the back and forth between Fourier transform, not Fourier transformed... Just keep calm and compute on. The definition tells us that the FOURIER TRANSFORM of the thing we want to know is

$$\hat{f}(\xi) \chi_{(-\alpha,\alpha)}(\xi).$$

We know that the Fourier transform of a convolution is a product of two Fourier transforms. Well, if we can find somebody whose Fourier transform is this weird $\chi_{(-\alpha,\alpha)}$, then we'll be in good shape. Let's try looking in BETA. We see on p. 320 that the Fourier transform of

$$\frac{\sin(\alpha t)}{\pi t}$$

is χ, as desired. Thus, we now know that

$$LP_\alpha(f)(x) = \int_{\mathbb{R}} f(x-t)\frac{\sin(\alpha t)}{\pi t}dt.$$

7. Lös problemet:

$$u_{tt} - u_{xx} = tx, \quad 0 < x < 4, \quad t \geq 0,$$

$$\begin{aligned} u(0,t) &= 20, \\ u_x(4,t) &= 0, \\ u(x,0) &= 20, \\ u_t(x,0) &= 0. \end{aligned}$$

(10 p)

Almost déjà vu right? Mais pas precisement... We have here an inhomogeneous wave equation. However, the inhomogeneity is *time dependent.* So, a steady state solution ain't gonna solve that problem. Next, we look at our boundary and initial conditions. The constant function, 20, satisfies that vertical list of conditions. So, we look for a function v to satisfy the inhomogeneous wave equation *but* with homogeneous BC and IC, thus we want v to satisfy

$$v_{tt} - v_{xx} = tx,$$

and $v(0,t) = v_x(4,t) = v(x,0) = v_t(x,0) = 0$. Our solution will be $u = 20 + v$. To solve the inhomogeneous heat equation, we will use the Fourier series method (Fourier series because on a bounded interval). The inhomogeneous part of the heat equation can be expressed using an L^2 OB $\{\phi_n\}$ for $[0,4]$ which satisfies the boundary condition and the SLP,

$$\phi_n''(x) + \lambda_n \phi_n(x) = 0, \quad \phi_n(0) = \phi_n'(4) = 0.$$

I leave it to you to check that the only λ_n for which there is such a $\phi_n \not\equiv 0$ are positive λ_n. The corresponding ϕ_n is thus a linear combination of sine and cosine, and to satisfy the BC at $x = 0$, we see that the cosine is out. So, we need a sine. In order to get the BC at $x = 4$, we need (up to multiplication by a factor which is constant with respect to x)

$$\phi_n = \sin((2n+1)\pi x/8), \quad \lambda_n = \frac{(2n+1)^2\pi^2}{64}.$$

Next, we shall allow the constant factor multiplying ϕ_n, to depend on time, and we write

$$v(t,x) = \sum_{n\in\mathbb{N}} c_n(t)\phi_n(x).$$

We can also express the xt side of the wave equation using the L^2 OB,

$$tx = t\sum_{n\geq 0} \widehat{x_n}\phi_n(x),$$

where

$$\widehat{x_n} = \frac{1}{2}\int_0^4 x\sin((2n+1/)\pi x/8)dx = \frac{8(-1)^n}{(n+1/2)^2}.$$

Next, we apply the wave operator to the expression for v in order to determine the unknown coefficient functions, c_n,

$$v_{tt} + v_{xx} = \sum_{n\geq 0} c_n''(t)\phi_n(x) - c_n(t)\phi_n''(x) = \sum_{n\geq 0} \left(c_n''(t) + \lambda_n c_n(t)\right)\phi_n(x).$$

We want this to equal

$$tx = \sum_{n\geq 0} t\widehat{x_n}\phi_n(x).$$

To obtain the equality, we equate the individual terms in each series, writing

$$(c_n''(t) + c_n(t)\lambda_n)\phi_n(x) = t\widehat{x_n}\phi_n(x).$$

Hence, we want c_n to satisfy the ODE:

$$c_n''(t) + \lambda_n c_n(t) = t\widehat{x_n}.$$

The homogeneous ODE

$$f'' + \lambda f = 0, \quad \lambda > 0, \quad \implies f(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$$

A particular solution to the inhomogeneous ODE is a function of the form

$$c(t) = at + b.$$

Substituting such a function into the ODE, we see that we need

$$c_n(t) = \frac{t\widehat{x_n}}{\lambda_n}.$$

Now we gotta look at the ICs. You see, the ϕ_n's take care of the BC's because we built them that way. However, they don't depend on time, so they can't help us with the ICs. We need the $c_n(t)$ to do that. Now, if we just take the particular solution to the ODE, we see that it vanishes at $t = 0$. However, we also want the derivative to vanish at $t = 0$, and it don't do that. So, we combine the particular solution with a solution to the homogeneous ODE. Hence, we want

$$c_n(t) = \frac{t\widehat{x_n}}{\lambda_n} + a_n\cos(\sqrt{\lambda_n}x) + b_n\sin(\sqrt{\lambda_n}x).$$

To make sure $c_n(0) = 0$ we need $a_n = 0$. To make sure $c_n'(0) = 0$, we need

$$\frac{\widehat{x_n}}{\lambda_n} + \sqrt{\lambda_n}b_n = 0 \implies b_n = -\frac{\widehat{x_n}}{\lambda_n^{3/2}}.$$

Hence, our full solution is

$$u(x,t) = 20 + \sum_{n\in\mathbb{N}} \left(\frac{t\widehat{x_n}}{\lambda_n} - \frac{\widehat{x_n}}{\lambda_n^{3/2}} \sin(\sqrt{\lambda_n}x) \right) \phi_n(x),$$

with

$$\lambda_n = \frac{(2n+1)^2\pi^2}{64}, \quad \widehat{x_n} = \frac{8(-1)^n}{(n+1/2)^2}, \quad \phi_n(x) = \sin((2n+1)\pi x/8).$$

8. Lös problemet:

$$u_t - u_{xx} - u_{yy} = 0, \quad -1 \le x, y \le 1, \quad t \ge 0,$$

med

$$\begin{aligned} u(-1,y,t) &= 25, \\ u(1,y,t) &= 25, \\ u(x,-1,t) &= 25, \\ u(x,1,t) &= 25, \\ u(x,y,0) &= (6-|x|)(6-|y|). \end{aligned}$$

(10 p)

The PDE is homogeneous. We would like to use Sturm-Liouville theory, but the BCs are not homogeneous. However, we quickly observe that a steady state solution, namely $u_0(x,y) = 25$. Then we look for $v(x,y,t)$ which vanishes on the boundary of the rectangle and has initial condition

$$v(x,y,0) = (6-|x|)(6-|y|) - 25.$$

The full solution shall be $v(x,y,t) + 25$.

To find v, we shall first separate all the variables, writing

$$v = TXY.$$

Then our equation becomes

$$T'(XY) - X''(TY) - Y''(TX) = 0 \iff \frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = \text{ constant } = \lambda.$$

Due to the fact that we have more information on X and Y, specifically

$$X(-1) = X(1) = 0, \quad Y(-1) = Y(1) = 0,$$

we consider them first. So, we have the equation

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \text{ constant } = \mu.$$

Thus, we have

$$X'' = \mu X, \quad X(-1) = X(1) = 0.$$

We see that there are non-zero solutions to this only if $\mu < 0$. In that case, X is a combination of sines and cosines. However, to get both boundary conditions, X can in fact be either a sine or a cosine. The two possibilities are (up to constant factors)

$$X(x) = \sin(n\pi x) \implies \mu = -n^2\pi^2,$$

and

$$X(x) = \cos((2m+1)\pi x/2) \implies \mu = (2m+1)^2\pi^2/4.$$

We compute the L^2 norm of these to be one, conveniently.

The equation for Y is

$$-\frac{Y''}{Y} = \mu - \lambda \implies \frac{Y''}{Y} = \lambda - \mu.$$

Similarly we have the possible solutions Y,

$$Y(y) = \sin(n\pi y) \implies \lambda - \mu = -n^2\pi^2 \implies \lambda = \mu - n^2\pi^2,$$

and

$$Y(x) = \cos((2m+1)\pi x/2) \implies \lambda - \mu = -(2m+1)^2\pi^2/4 \implies \lambda = \mu - (2m+1)^2\pi^2/4.$$

Thus, we have the full set of solutions which consists of products of

$$\sin(n\pi x),\ \cos((2n+1)\pi x/2),\ \text{together with}\ \sin(m\pi y),\ \cos((2m+1)\pi y/2).$$

The corresponding λs are

$$-(n^2\pi^2 + m^2\pi^2),\ -\left(n^2\pi^2 + \frac{(2m+1)^2\pi^2}{4}\right),$$

and

$$-\left(\frac{(2n+1)^2\pi^2}{4} + m^2\pi^2\right),\ -\left(\frac{(2n+1)^2\pi^2}{4} + \frac{(2m+1)^2\pi^2}{4}\right).$$

Then, we have (up to constant factors)

$$T(t) = e^{\lambda t}.$$

Next, we determine said constant factors, by writing

$$v(t,x,y) = \sum_{\lambda} c_\lambda e^{\lambda t} X_\lambda(x) Y_\lambda(y),$$

where the sum is over all λ given above. The initial condition says that

$$v(0,x,y) = \sum_{\lambda} c_\lambda X_\lambda(x) Y_\lambda(y) = (6-|x|)(6-|y|) - 25.$$

Hence, the coefficients c_λ come from the Fourier coefficients of $(6-|x|)(6-|y|)-25$. We observe that this is an *even* function. Thus for any odd function $X_\lambda(x)$, we have

$$\int_{-1}^{1} ((6-|x|)(6-|y|)-25)X_\lambda(x)dx = 0,$$

and similarly, for an odd function $Y_\lambda(y)$,

$$\int_{-1}^{1} ((6-|x|)(6-|y|)-25)Y_\lambda(y)dy = 0.$$

Thus, the only non-zero Fourier coefficients come from the cosine terms. These coefficients are given by

$$c_{m,n} = \int_{-1}^{1}\int_{-1}^{1} ((6-|x|)(6-|y|)-25)\cos((2m+1)\pi x/2)\cos((2n+1)\pi y/2)dxdy.$$

The corresponding

$$\lambda_{m,n} = -\frac{(2m+1)^2\pi^2 + (2n+1)^2\pi^2}{4}.$$

Hence,

$$v(t,x,y) = \sum_{m,n\in\mathbb{N}} c_{m,n} e^{\lambda_{m,n}t}\cos((2m+1)\pi x/2)\cos((2n+1)\pi y/2),$$

and the full solution is

$$u(t,x,y) = 25 + v(t,x,y).$$

Comment regarding grading and partial credit: If your answer is *wrong,* but you received some partial credit, you're welcome. That's because a wrong answer is, strictly speaking, worth nothing. What happens if you solve the heat equation wrong, use that erroneous solution at work to, say create a rocket to be sent to space with some astronauts inside? Your solution was really worth a lot of partial credit if it results in a ruined rocket and some dead astronauts. In the case of PDEs, there's also no excuse. A PDE is an equation. So, you can always PLUG your solution in to the equation and check whether it solves the equation or not... So, please don't whinge for more partial credit for wrong solutions, because this may end up having the result that the entire concept of partial credit disappears for future generations, and they probably don't want that. Just know that all exams shall be graded by the same rules, because fairness is the fundamental theorem of grading. These rules are slightly difficult to articulate precisely, because people find so very many and creative ways to go wrong. It's all about giving equal points for equal progress (or lack thereof) on each problem. Each problem is graded across all exams before moving on to the next one, to try to ensure fairness (i.e. #1 on all exams graded, to make sure all the #1s are graded the same way, then proceeding to #2 on all exams, etc).

Errors do happen on some occasions though, and if you feel there may have been an error (it is helpful to compare with classmates), please let me know, and I will fix it! ♡ Julie Rowlett