MVE 290/030 PROOFS OF THE THEORY QUESTIONS

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(1) Proof of pointwise convergence of Fourier series (Theorem 2.1 of Folland).
(2) Proof of the formula for the relationship between the Fourier coefficients for a function and its derivative (Theorem 2.2 of Folland).
(3) Proof of Theorem 7.3 in Folland.
(4) The Fourier inversion formula.
(5) Plancharel's Theorem.
(6) Proof of the Sampling Theorem.
(7) Proof of Theorem 3.4.
(8) Proof of Theorem 3.8 on the best approximation.
(9) Proof of Theorem 3.9 (a) and (b).
(10) Proof of the Generating Function for $J_n(x)$, formula (5.20) in Folland.
(11) Proof of the orthogonality of the Hermite polynomials (this is part of the proof of Theorem 6.11 in Folland).
(12) Proof of Theorem 6.13, that is to derive the generating formula for the Hermite polynomials (6.35).

1. Pointwise convergence of Fourier series for continuous, piecewise $\mathcal{C}^1$ functions

This is a big theorem. The statement we shall prove is the following

Theorem 1.1. *Let f be a 2π periodic function. Assume that f is piecewise continuous on $\mathbb{R}$, and that for every $x \in \mathbb{R}$, the left and right limits of both f and f' exist at x, and these are finite. Let*

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Then

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right), \quad \forall x \in \mathbb{R}.$$

Proof. The result should hold for each and every point $x \in \mathbb{R}$. So, first, we fix a point $x \in \mathbb{R}$. Next, as usual, we should use the definitions, so we expand the series using its definition. So, we write

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx}.$$

Now, let's move that lonely e^{inx} inside the integral so it can get close to its friend, e^{-iny}. Then,

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny+inx} dy.$$

OBS that f on the right is not involved with x, but in the theorem we are trying to prove, we want to relate $S_N(x)$ to $f(x)$. How can we get an x inside the f? Simple, we change the variable. Let $t = y - x$. Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt.$$

Remember that very first fact we proved for periodic functions? It said that the integral of a periodic function of period P from any point a to $a+P$ is the same, no matter what a is. Here $P = 2\pi$. Hence

$$\int_{-\pi-x}^{\pi-x} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x)e^{-int}dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x)\frac{1}{2\pi}\sum_{-N}^{N} e^{int}dt.$$

This is how we get to the N^{th} Dirichlet kernel. Let

$$D_N(t) = \frac{1}{2\pi}\sum_{-N}^{N} e^{int}.$$

We now shall make two observations about this function. First, if we collect the even and odd terms, recall that

$$n \in \mathbb{N} \implies e^{int} + e^{-int} = 2\cos(nt), n > 0.$$

Hence, we can pair up all the terms ± 1, ± 2, etc, and write

$$D_N(t) = \frac{1}{2\pi} + \sum_{n=1}^{N} \frac{1}{\pi}\cos(nt).$$

This leads to our first observation about $D_N(t)$, that is its integral

$$\int_{-\pi}^{\pi} D_N(t)dt = 1.$$

Moreover, this also shows that $D_N(t)$ is an even function, hence

$$\int_{-\pi}^{0} D_N(t)dt = \frac{1}{2} = \int_{0}^{\pi} D_N(t)dt. \tag{1.1}$$

The second observation is that $D_N(t)$ looks almost like a geometric series, the problem is that it goes from minus exponents to positive ones. We can fix that right up by factoring out the largest negative exponent, so

$$D_N(t) = \frac{1}{2\pi}e^{-iNt}\sum_{n=0}^{2N} e^{int}.$$

We know how to sum a partial geometric series, don't we? This gives

$$D_N(t) = \frac{1}{2\pi}e^{-iNt}\frac{1-e^{i(2N+1)t}}{1-e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}.$$

Now we return to our problem, in which we have

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x)D_N(t)dt.$$

We want to show that $S_N(x)$ converges to the average of the right and left hand limits of f. In other words, this is equivalent to showing that

$$\lim_{N\to\infty}\left|S_N(x) - \frac{1}{2}\left(f(x_-) + f(x_+)\right)\right| = 0.$$

The S_N business has an integral, but the $f(x_\pm)$ don't. They have got a convenient factor of one half, so we use (1.1) to exploit this

$$\frac{1}{2}f(x_-) = \int_{-\pi}^{0} D_N(t)dt f(x_-), \quad \frac{1}{2}f(x_+) = \frac{1}{2} = \int_{0}^{\pi} D_N(t)dt f(x_+).$$

Hence we are bound to prove that

$$\lim_{N\to\infty}\left|S_N(x) - \int_{-\pi}^{0} D_N(t)f(x_-)dt - \int_{0}^{\pi} D_N(t)f(x_+)dt\right| = 0.$$

Now, we use that

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt.$$

Hence, we want to show that

$$\left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \int_{-\pi}^{0} D_N(t) f(x_-) dt - \int_0^{\pi} D_N(t) f(x_+) dt \right| \to 0, \text{ as } N \to \infty.$$

It is quite natural to split things into two parts

$$\left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_0^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right|.$$

Now, we know we've got to use the second expression for $D_N(t)$, and here's where it will come in handy. Let's insert it

$$\left| \int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_-)) dt + \int_0^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_+)) dt \right|.$$

Now, we know that if we take a function which is bounded, then its Fourier coefficients tend to 0, meaning $c_n \to 0$ as $|n| \to \infty$. We've got those e^{-iNt} and $e^{i(N+1)t}$ which look a lot like part of the definition of Fourier coefficient c_n for $|n|$ large... However, we've got this integrand defined two different ways on either side of zero. So, let's just make a try for something and define a new function

$$g(t) = \frac{f(t+x) - f(x_-)}{1 - e^{it}}, \quad \text{for } t < 0,$$

$$g(t) = \frac{f(t+x) - f(x_+)}{1 - e^{it}}, \quad \text{for } t > 0.$$

How to define this function at zero? Let's look at the limit

$$\lim_{t \to 0_-} \frac{f(t+x) - f(x_-)}{1 - e^{it}} = \lim_{t \to 0_-} \frac{t(f(t+x) - f(x_-))}{t(1 - e^{it})} = \frac{f'(x_-)}{-ie^{i0}} = if'(x_-).$$

For the other side, a similar argument shows that

$$\lim_{t \to 0_+} \frac{f(t+x) - f(x_-)}{1 - e^{it}} = if'(x_+).$$

So, depending upon whether $f'(x_-) = f'(x_+)$ or not, the function g will be continuous at 0, or not. However, even if it's not continuous, it is at least piecewise continuous, as well as piecewise differentiable, just like the original function f is. To see this, we see that for all other points $t \in [-\pi, \pi]$, the denominator of g is non-zero, and the numerator has the same properties as f. Therefore the above shows that g is indeed quite a lovely function on $[-\pi, \pi]$. The most important fact is that it is bounded on the closed interval $[-\pi, \pi]$, and hence its Fourier coefficients tend to zero by Bessel's inequality. This follows from the fact that any bounded function on a bounded interval, like $[-\pi, \pi]$, is in L^2 on that interval, i.e. in $L^2([-\pi, \pi])$.

Hence, we are looking at

$$\lim_{N \to \infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-iNt} dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(-N-1)t} g(t) dt \right| = \lim_{N \to \infty} |c_N(g) - c_{-N-1}(g)|,$$

where above, $c_N(g)$ is the N^{th} Fourier coefficient of g,

$$c_N(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-iNt} dt,$$

and similarly, $c_{-N-1}(g)$ is the $-N-1^{st}$ Fourier coefficient of g,

$$c_{-N-1}(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(-N-1)t} g(t) dt.$$

By Bessel's inequality,

$$c_N(g) \to 0 \text{ as } N \to \infty, \quad \text{and } c_{-N-1}(g) \to 0 \text{ as } N \to \infty.$$

Hence

$$\lim_{N\to\infty}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-iNt}dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(-N-1)t}g(t)dt\right| = |0+0| = 0.$$

□

2. The Fourier coefficients of a function and its derivative

The nickname for this theory item is do NOT differentiate the series termwise!!! Sure, there is a result in the text later on which says that a function satisfying these hypotheses has a Fourier series which converges absolutely and uniformly, but do you know how to prove that? You use this result. Hence, if you try to use that result to prove this one, you've just run around in a circle and proven nothing. If you wanted to go down that road - correctly - using termwise differentiation of the Fourier series of f, you'd need to prove the absolute, uniform convergence by some independent means. I do not recommend this. This looks hard. As you will see, the proof below is pleasantly elementary. So, why make things hard and complicated?

Theorem 2.1. *This time in Swedish for fun! Låt f vara en 2π-periodisk funktion med $f \in C^2(\mathbb{R})$. Sedan Fourierkoefficienterna c_n av f och Fourierkoefficienterna c_n' av f' uppfyller*

$$c_n' = inc_n.$$

Proof. We quite simply use the definitions of the Fourier series and coefficients of f and f' respectively. By the hypothesis,

$$f(x) = \sum_{\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

and

$$f'(x) = \sum_{\mathbb{Z}} c_n' e^{inx}, \quad c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx.$$

Integrating by parts and using the periodicity of f and consequently also f' as well as the periodicity of e^{-inx} we have

$$c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} -f(x)(-in)e^{-inx}dx = inc_n.$$

□

3. The big bad convolution approximation theorem

This theory item is Theorem 7.3, regarding approximation of a function by convoluting it with a so-called "approximate identity." This theorem and its proof are both rather long. The proof relies very heavily on knowing the definition of limits and how to work with those definitions, so if you're not comfortable with ϵ and δ style arguments, it would be advisable to brush up on these.

Theorem 3.1. *Let $g \in L^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} g(x)dx = 1.$$

Define

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\epsilon > 0$,

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon\to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof. We would like to show that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(x-y)g_\epsilon(y)dy = \alpha f(x+) + \beta f(x-)$$

which is equivalent to showing that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(x-y)g_\epsilon(y)dy - \alpha f(x+) - \beta f(x-) = 0.$$

We now insert the definitions of α and β, so we want to show that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(x-y)g_\epsilon(y)dy - \int_{-\infty}^0 f(x+)g(y)dy - \int_0^\infty f(x-)g(y)dy = 0.$$

We can prove this if we show that

$$\lim_{\epsilon\to 0}\int_{-\infty} f(x-y)g_\epsilon(y)dy - \int_{-\infty}^0 f(x+)g(y)dy = 0$$

and also

$$\lim_{\epsilon\to 0}\int_0^\infty f(x-y)g_\epsilon(y)dy - \int_0^\infty f(x-)g(y)dy = 0.$$

In the textbook, Folland proves that the second of these holds. So, for the sake of diversity, we prove that the first of these holds. The argument is the same for both, so proving one of them is sufficient.

Hence, we would like to show that by choosing ϵ sufficiently small, we can make

$$\int_{-\infty}^0 f(x-y)g_\epsilon(y)dy - \int_{-\infty}^0 f(x+)g(y)dy$$

as small as we like. To make this precise, let us assume that "as small as we like" is quantified by a very small $\delta > 0$. This is part of the whole points of limits. This quantity,

$$\int_{-\infty}^0 f(x-y)g_\epsilon(y)dy - \int_{-\infty}^0 f(x+)g(y)dy$$

can be made arbitrarily small by choosing ϵ sufficiently small, but it may never actually vanish as long as $\epsilon > 0$. You'll just need to sit for a while and think about that, to make sure you're really comfortable with the concept of limits...

Proceeding with the proof, we smash the two integrals together, writing

$$\int_{-\infty}^0 \left(f(x-y)g_\epsilon(y) - f(x+)g(y)\right) dy.$$

Well, this is a bit inconvenient, because in the first part we have g_ϵ, but in the second part it's just g. So, we make a small observation,

$$\int_{-\infty}^0 g(y)dy = \int_{-\infty}^0 g(z/\epsilon)\frac{dz}{\epsilon} = \int_{-\infty}^0 g_\epsilon(z)dz$$

Above, we have made the substitution $z = \epsilon y$, so $y = z/\epsilon$, and $dz/\epsilon = dy$. The limits of integration don't change. By this calculation,

$$\int_{-\infty}^0 f(x+)g(y)dy = \int_{-\infty}^0 f(x+)g_\epsilon(y)dy.$$

(Above the integration variable was called z, but what's in a name? The name of the integration variable doesn't matter!). Moreover, note that $f(x+)$ is a constant, so it's just sitting there doing nothing. Hence, we have computed that

$$\int_{-\infty}^0 \left(f(x-y)g_\epsilon(y) - f(x+)g(y)\right) dy = \int_{-\infty}^0 g_\epsilon(y)\left(f(x-y) - f(x+)\right) dy.$$

Remember that $y \le 0$ where we're integration. Therefore, $x - y \ge x$. Moreover, by definition

$$\lim_{y\uparrow 0} f(x-y) = f(x+) \implies \lim_{y\uparrow 0} f(x-y) - f(x+) = 0.$$

By definition of limit (if you're not comfortable with this definition by now, you really need to get comfortable with it!) there exists $y_0 < 0$ such that for all $y \in (y_0, 0)$

$$|f(x-y) - f(x+)| < \widetilde{\delta}.$$

We are using $\widetilde{\delta}$ for now, to indicate that $\widetilde{\delta}$ is going to be something in terms of δ, engineered in such a way that at the end of our argument we get that for ϵ sufficiently small,

$$\left| \int_{-\infty}^{0} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy \right| < \delta.$$

So, to figure out this $\widetilde{\delta}$, we use our estimate on the part of the integral from y_0 to 0,

$$\left| \int_{y_0}^{0} (f(x-y) - f(x+)) g_\epsilon(y) dy \right| \leq \int_{y_0}^{0} |f(x-y) - f(x+)||g_\epsilon(y)| dy$$

$$\leq \widetilde{\delta} \int_{y_0}^{0} |g_\epsilon(y)| dy \leq \widetilde{\delta} \int_{\mathbb{R}} |g_\epsilon(y)| dy = \widetilde{\delta} ||g||.$$

Above, we have used the same substitution trick to see that

$$\int_{\mathbb{R}} |g_\epsilon(y)| dy = \int_{\mathbb{R}} |g(z)| dz = ||g||,$$

where $||g||$ is the $L^1(\mathbb{R})$ norm of g. By assumption, $g \in L^1(\mathbb{R})$, so this L^1 norm is finite. Moreover, because we know that

$$\int_{\mathbb{R}} g(y) dy = 1,$$

we know that

$$||g|| = \int_{\mathbb{R}} |g(y)| dy \geq \left| \int_{\mathbb{R}} g(y) dy \right| = 1.$$

Hence, I propose setting

$$\widetilde{\delta} = \frac{\delta}{2||g||}.$$

Note that we're not dividing by zero, by the above observation that $||g|| \geq 1$. So, this is a perfectly decent number. Then, we have the estimate (repeating the above estimate)

$$\left| \int_{y_0}^{0} (f(x-y) - f(x+)) g_\epsilon(y) dy \right| \leq \int_{y_0}^{0} |f(x-y) - f(x+)||g_\epsilon(y)| dy$$

$$\leq \widetilde{\delta} \int_{y_0}^{0} |g_\epsilon(y)| dy \leq \widetilde{\delta} \int_{\mathbb{R}} |g_\epsilon(y)| dy = \widetilde{\delta} ||g|| = \frac{\delta}{2}.$$

To complete the proof, we just need to estimate the other part of the integral, from $-\infty$ to y_0. It is important to remember that

$$y_0 < 0.$$

So, we wish to estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+)) g_\epsilon(y) dy \right|.$$

Here we need to consider the two possible cases given in the statement of the theorem separately. First, let us assume that f is bounded, which means that there exists $M > 0$ such that $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$. Hence

$$|f(x-y) - f(x+)| \leq |f(x-y)| + |f(x+)| \leq 2M.$$

So, we have the estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+)) g_\epsilon(y) dy \right| \leq \int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\epsilon(y)| dy \leq 2M \int_{-\infty}^{y_0} |g_\epsilon(y)| dy.$$

We shall do a substitution now, letting $z = y/\epsilon$. Then, as we have computed before,

$$\int_{-\infty}^{y_0} |g_\epsilon(y)| dy = \int_{-\infty}^{y_0/\epsilon} |g(z)| dz.$$

Here the limits of integration do change, because $y_0 < 0$. Specifically $y_0 \neq 0$, which is why the top limit changes. Now, let's think about what happens as $\epsilon \to 0$. We're integrating between $-\infty$ and y_0/ϵ. We know that $y_0 < 0$. So, when we divide it by a really small, but still positive number, like ϵ, then $y_0/\epsilon \to -\infty$ as $\epsilon \to 0$. Moreover, we know that

$$\int_{-\infty}^{0} |g(y)|dy < \infty.$$

What this really means is that

$$\lim_{R\to-\infty} \int_{R}^{0} |g(y)|dy = \int_{-\infty}^{0} |g(y)|dy < \infty.$$

Hence,

$$\lim_{R\to-\infty} \int_{-\infty}^{0} |g(y)|dy - \int_{R}^{0} |g(y)|dy = 0.$$

Of course, we know what happens when we subtract the integral, which shows that

$$\lim_{R\to-\infty} \int_{-\infty}^{R} |g(y)|dy = 0.$$

Since

$$\lim_{\epsilon\to 0} y_0/\epsilon = -\infty,$$

this shows that

$$\lim_{\epsilon\to 0} \int_{-\infty}^{y_0/\epsilon} |g(y)|dy = 0.$$

Hence, by definition of limit (see, here it comes again), there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$,

$$\int_{-\infty}^{y_0/\epsilon} |g(y)|dy < \frac{\delta}{4(M+1)}.$$

Then, combining this with our estimates, above, which we repeat here,

$$\left|\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\epsilon(y)dy\right| \le \int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\epsilon(y)|dy \le 2M \int_{-\infty}^{y_0} |g_\epsilon(y)|dy$$

$$< 2M\frac{\delta}{4(M+1)} < \frac{\delta}{2}.$$

Therefore, we have the estimate that for all $\epsilon \in (0, \epsilon_0)$,

$$\left|\int_{-\infty}^{0} g_\epsilon(y)\,(f(x-y) - f(x+))\,dy\right|$$

$$\le \int_{-\infty}^{0} |g_\epsilon(y)||f(x-y)-f(x+)|dy \le \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\epsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\epsilon(y)|dy$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Finally, we consider the other case in the theorem, which is that g vanishes outside a bounded interval. We retain the first part of our estimate, that is

$$\int_{y_0}^{0} |f(x-y) - f(x+)||g_\epsilon(y)|dy < \frac{\delta}{2}.$$

Next, we again observe that

$$\lim_{\epsilon\downarrow 0} \frac{y_0}{\epsilon} = -\infty.$$

By assumption, we know that there exists some $R > 0$ such that

$$g(x) = 0 \forall x \in \mathbb{R} \text{ with } |x| > R.$$

Hence, we may choose ϵ sufficient small so that

$$\frac{y_0}{\epsilon} < -R.$$

Specifically, let

$$\epsilon_0 = \frac{y_0}{-R} > 0.$$

Then for all $\epsilon \in (0, \epsilon_0)$ we compute that

$$\frac{y_0}{\epsilon} < -R.$$

Hence for all $y \in (-\infty, y_0/\epsilon)$ we have $g(y) = 0$. Thus, we compute as before using the substitution $z = y/\epsilon$,

$$\int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\epsilon(y)|dy = \int_{-\infty}^{y_0/\epsilon} |f(x - \epsilon z) - f(x+)||g(z)|dz = 0,$$

because $g(z) = 0 \forall z \in (-\infty, y_0/\epsilon)$. Thus, we have the total estimate that for all $\epsilon \in (0, \epsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\epsilon(y) \left(f(x-y) - f(x+)\right) dy \right|$$

$$\leq \int_{-\infty}^{0} |g_\epsilon(y)||f(x-y)-f(x+)|dy \leq \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\epsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\epsilon(y)|dy$$

$$< 0 + \frac{\delta}{2} \leq \delta.$$

□

4. The Fourier inversion formula

This theory item is really a julklapp. All one must know is the Fourier inversion formula.

Theorem 4.1 (FIT)**.** *Assume that $f \in L^2(\mathbb{R})$.*

(1) If we define the Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(y) e^{-2\pi i y \xi} dy,$$

and the inverse Fourier transform by

$$\check{f}(\xi) = \hat{f}(-\xi),$$

we have that

$$\check{(\hat{f})} = f.$$

(2) Alternatively, if we define the Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(y) e^{-iy\xi} dy,$$

and the inverse Fourier transform by

$$\check{f}(\xi) = \hat{f}(-\xi),$$

then we have that

$$\check{(\hat{f})} = 2\pi f.$$

Remark. For this theory item, you only need to give the formula for one of the two cases. So, please pick your favorite definition of Fourier transform, either with or without the 2π, and learn the FIT for your favorite definition of the Fourier transform.

5. Plancharel's Theorem

This is also a rather simple one.

Theorem 5.1. *Assume $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$.*

(1) *In case the Fourier transform is defined as*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx,$$

then we have

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle,$$

and

$$||\hat{f}||^2 = ||f||^2.$$

(2) *In case the Fourier transform is defined via,*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,$$

then we have

$$\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle,$$

and

$$||\hat{f}||^2 = 2\pi ||f||^2.$$

Remark. You only need to learn the theorem for one of the two cases. So, using your favorite definition of Fourier transform, please learn the theorem and its proof!

Proof. There is one idea which is key here, and that is to start on the right side. Why? Because it is easier, at least it is easier for me. When I try starting on the left side, it gets very messy very quickly. So, better not to do that.

So, we will start with the right side, that is the side with the usual inner products (not the inner products of the Fourier transforms). It is a bit easier to prove the formulas for the first definition of the Fourier transform, because the inversion formula is simpler in that we don't need to remember where to stick the 2π in the inversion formula. So, we will first prove for the 2π version of the Fourier transform.

We write

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

By the Fourier inversion theorem,

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi) d\xi.$$

We insert this,

$$\langle f, g \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi) \overline{g(x)} d\xi dx.$$

Next, we consider

$$\int_{\mathbb{R}} e^{2\pi i x \xi} \overline{g(x)} dx.$$

We have two things in there, one is the complex conjugate of g, and the other is the $e^{2\pi i x \xi}$. We somehow want to get $\hat{g}$ coming out, but the sign is wrong on the complex exponential. However, we can fix this by observing that

$$\overline{e^{-2\pi i x \xi}} = e^{2\pi i x \xi}.$$

Thus,

$$\int_{\mathbb{R}} e^{2\pi i x \xi} \overline{g(x)} dx = \int_{\mathbb{R}} \overline{e^{-2\pi i x \xi} g(x)} dx = \overline{\int_{\mathbb{R}} e^{-2\pi i x \xi} g(x) dx} = \overline{\hat{g}(\xi)}.$$

We have therefore computed that

$$\langle f, g \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi) \overline{g(x)} d\xi dx = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \langle \hat{f}, \hat{g} \rangle.$$

Setting $g = f$, this immediately implies

$$||f||^2 = ||\hat{f}||^2.$$

Above, this is of course the $L^2(\mathbb{R})$ norm!

Next, we shall use what we have just done to prove the formula for the non-2π Fourier transform. Let us denote

$$\tilde{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

Then, $\hat{f}(\xi) = \tilde{f}(2\pi\xi)$. So, we have

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle = \int_{\mathbb{R}} \tilde{f}(2\pi\xi) \overline{\tilde{g}(2\pi\xi)} d\xi.$$

Let $y = 2\pi\xi$, then $dy = 2\pi d\xi$, and $(2\pi)^{-1} dy = d\xi$, so we have

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{f}(y) \overline{\tilde{g}(y)} dy = \frac{1}{2\pi} \langle \tilde{f}, \tilde{g} \rangle.$$

Multiplying by 2π gives

$$\langle \tilde{f}, \tilde{g} \rangle = 2\pi \langle f, g \rangle.$$

Finally, setting $g = f$ shows that

$$||\tilde{f}||^2 = 2\pi ||f||^2.$$

For the sake of completeness, we also include an independent proof of the non-2π Fourier transform. The idea is precisely the same, start on the right side with the usual L^2 inner product. Thus, we consider

$$\int_{\mathbb{R}} f(y) \overline{g(y)} dy.$$

Next, we use the FIT for this case to write

$$f(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyx} \tilde{f}(x) dx.$$

Substituting,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iyx} \tilde{f}(x) \overline{g(y)} dy dx.$$

Next, we observe that we've got something very close to the Fourier transform of g sitting there,

$$\int_{\mathbb{R}} e^{iyx} \overline{g(y)} dy.$$

This isn't quite the Fourier transform, because the sign of the exponential is wrong. However, observe that

$$\overline{e^{-iyx}} = e^{iyx},$$

so

$$\int_{\mathbb{R}} e^{iyx} \overline{g(y)} dy = \int_{\mathbb{R}} \overline{e^{-iyx} g(y)} dy = \overline{\int_{\mathbb{R}} e^{-iyx} g(y) dy} = \overline{\tilde{g}(x)}.$$

Thus, we have computed that

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{f}(x) \overline{\tilde{g}(x)} dx = \frac{1}{2\pi} \langle \tilde{f}, \tilde{g} \rangle.$$

Moving the 2π around gives us

$$2\pi \langle f, g \rangle = \langle \tilde{f}, \tilde{g} \rangle.$$

Setting $f = g$ immediately also gives

$$2\pi ||f||^2 = ||\tilde{f}||^2.$$

□

6. The Sampling Theorem

Theorem 6.1. *Let $f \in L^2(\mathbb{R})$. This theory item is to prove either item (1) or item (2) below.*

(1) If we take the definition of the Fourier transform of f to be

$$\int_{\mathbb{R}} e^{-ix\xi} f(x)dx,$$

and we then assume that there is $L > 0$ so that $\hat{f}(\xi) = 0 \ \forall \xi \in \mathbb{R}$ with $|\xi| > L$, then:

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

(2) If we take the definition of the Fourier transform of f to be

$$\int_{\mathbb{R}} e^{-2\pi i \xi x} f(x)dx,$$

then we have

$$f(t) = \sum_{n \in Z} f\left(\frac{n}{2L}\right) \frac{\sin(n\pi - t2\pi L)}{n\pi - 2\pi L t}.$$

Remark. Taking either version above, we note that for any $M \geq L$, the same result holds replacing L with M. We shall see that in this way, although the two statements appear different, there is no contradiction. The reason the same result holds for L replaced with M, as long as $M \geq L$, is because if we assume the Fourier transform vanishes for $|\xi| > L$, then it also vanishes for $|\xi| > M$ if $M \geq L$. Hence, the statement also holds with L replaced by M.

Proof. For this theory item, you only need to prove the result for one of the versions of the Fourier transform, either without the 2π or with the 2π. You are free to choose which version.

First, we shall prove the statement of the theorem for the Fourier transform defined without the 2π. This theorem is all about the interaction between Fourier series and Fourier coefficients and how to work with both simultaneously. Since the Fourier transform $\hat{f}$ has compact support, the following equality holds as elements of $L^2([-L, L])$,

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x)dx.$$

We use the Fourier inversion theorem (FIT) to write

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixt} \hat{f}(x)dx = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \hat{f}(x)dx.$$

On the left we have used the fact that $\hat{f}$ is supported in the interval $[-L, L]$, thus the integrand is zero outside of this interval, so we can throw that part of the integral away.

We next substitute the Fourier expansion of $\hat{f}$ into this integral,

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} c_n e^{in\pi x/L} dx.$$

Let us take a closer look at the coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x)dx = \frac{1}{2L} \int_{\mathbb{R}} e^{ix(-n\pi/L)} \hat{f}(x)dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

In the second equality we have used the fact that $\hat{f}(x) = 0$ for $|x| > L$, so by including that part we don't change the integral. In the third equality we have used the FIT!!! So, we now substitute this into our formula above for

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L} dx$$

This is approaching the form we wish to have in the theorem, but the argument of the function f has a pesky negative sign. That can be remedied by switching the order of summation, which does not change the sum, so

$$f(t) = \frac{1}{2L} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{-in\pi x/L} dx.$$

We may also interchange the summation with the integral[i]

$$f(t) = \frac{1}{2L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \int_{-L}^{L} e^{x(it-in\pi/L)} dx.$$

We then compute

$$\int_{-L}^{L} e^{x(it-in\pi/L)} dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)} \sin(Lt-n\pi).$$

Substituting,

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \frac{\sin(Lt-n\pi)}{Lt-n\pi}.$$

Next, we observe how things change depending upon our definition of the Fourier transform. In the first statement of the theorem, we are assuming that the Fourier transform vanishes outside of $[-L, L]$. Now, let us use as we did in the proof of Plancharel's theorem

$$\hat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i\xi x} f(x) dx, \quad \tilde{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx.$$

If we assume that $\hat{f}$ vanishes outside of $[-L, L]$, then $\tilde{f}(\xi) = \hat{f}(\xi/2\pi)$ vanishes outside of $[-2\pi L, 2\pi L]$. Hence, observing that $M = 2\pi L > L$, we have by the first version of the theorem together with the remark immediately following the theorem that

$$f(t) = \sum_{n\in Z} f\left(\frac{n\pi}{M}\right) \frac{\sin(n\pi - tM)}{n\pi - tM} = \sum_{n\in Z} f\left(\frac{n}{2L}\right) \frac{\sin(n\pi - t2\pi L)}{n\pi - 2\pi Lt}.$$

This is the reason why the result we proved in lecture was the statement above, because we were using the 2π version of the Fourier transform. However, by the remark, there is no contradiction here.

Now, we shall also provide that proof, for the sake of completeness, for the second version of the theorem. We take the definition of the Fourier transform with the 2π. By the Fourier inversion formula in this case,

$$f(t) = \int e^{2\pi it\xi} \hat{f}(\xi) d\xi = \int_{-L}^{L} e^{2\pi i\xi t} \hat{f}(\xi) d\xi,$$

because we've assumed that $\hat{f}(\xi) = 0$ for $|\xi| > L$. We expand $\hat{f}$ as a Fourier series, writing

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{in\pi\xi/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} \hat{f}(y) e^{-in\pi y/L} dy,$$

but we note that due to the vanishing of $\hat{f}$ outside of $[-L, L]$ we have

$$c_n = \frac{1}{2L} \int_{\mathbb{R}} \hat{f}(y) e^{-in\pi y/L} dy.$$

Then, we observe by the FIT that

$$c_n = \frac{1}{2L} f\left(-\frac{n}{2L}\right).$$

So, we substitute the series into the FIT formula for f, with

$$f(t) = \int e^{2\pi it\xi} \hat{f}(\xi) d\xi = \int_{-L}^{L} e^{2\pi it\xi} \hat{f}(\xi) d\xi = \int_{-L}^{L} e^{2\pi i\xi t} \sum_{n\in\mathbb{Z}} e^{in\pi\xi/L} \frac{1}{2L} f\left(-\frac{n}{2L}\right).$$

[i]None of this makes sense pointwise; we are working over L^2. The key property which allows interchange of limits, integrals, sums, derivatives, etc is *absolute convergence*. This is the case here because elements of L^2 have $\int |f|^2 < \infty$. That is precisely the type of absolute convergence required.

Re-writing things a bit,

$$f(t) = \frac{1}{2L} \sum_{n\in\mathbb{Z}} \int_{-L}^{L} e^{i\pi\xi(2t+n/L)} f\left(-\frac{n}{2L}\right) d\xi.$$

We compute the integral,

$$\int_{-L}^{L} e^{i\pi\xi(2t+n/L)} d\xi = \frac{e^{i\pi L(2t+n/L)} - e^{-i\pi L(2t+n/L)}}{i\pi(2t+n/L)} = \frac{2\sin(\pi 2tL + \pi n)}{\pi(2t+n/L)}.$$

Substituting back,

$$f(t) = \frac{1}{2L} \sum_{n\in\mathbb{Z}} \frac{2\sin(\pi 2tL + \pi n)}{\pi(2t+n/L)} f\left(-\frac{n}{2L}\right).$$

Finally, we note that because $n \in \mathbb{Z}$, we can swap n and $-n$ above, which just amounts to summing in a different order, but because the sum converges absolutely, uniformly, beautifully, we may do this, so,

$$f(t) = \frac{1}{2L} \sum_{n\in\mathbb{Z}} \frac{2\sin(\pi 2tL - \pi n)}{\pi(2t-n/L)} f\left(\frac{n}{2L}\right) = \sum_{n\in\mathbb{Z}} \frac{\sin(\pi 2tL - \pi n)}{2t\pi L - n\pi} f\left(\frac{n}{2L}\right).$$

Using the fact that sin is an odd function, this is also equal to

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2L}\right) \frac{\sin(\pi n - 2\pi tL)}{\pi n - 2\pi tL}.$$

This proof may seem super long, but that is because we have proven it two different, but equivalent ways. You just need to pick one version and learn that proof, so you've only got to do half the work of what we've done here. □

7. Proof of the 3 equivalent conditions to be an ONB in a Hilbert space

This seems to be a fun one for some reason. It is rather nicely straightforward. Perhaps what makes it so nice is the pleasant setting of a Hilbert space, or translated directly from German, a Hilbert room. Hilbert rooms are cozy.

Theorem 7.1. *Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum, H. Följande tre är ekvivalenta:*

$$(1) \quad f \in H \text{ och } \langle f, \phi_n \rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2.$$

Proof. We shall proceed in order prove (1) $\implies$ (2), then (2) $\implies$ (3), and finally (3) $\implies$ (1). Just stay calm and carry on. So we begin by assuming (1) holds, and then we shall show that (2) must hold as well. First, we note that by Bessel's inequality, the series

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 \le ||f||^2 < \infty.$$

Hence, if we know anything about convergent series, then we sure better know that the tail of the series tends to zero. The tail of the series is

$$\sum_{n\ge N} |\langle f, \phi_n \rangle|^2 \to 0 \text{ as } N \to \infty.$$

Now, let us define some elements in our Hilbert space, which we shall show comprise a Cauchy sequence. Let

$$g_N := \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n.$$

For $M \geq N$, we have, using the Pythagorean Theorem and the orthonormality of the $\{\phi_n\}$,

$$||g_M - g_N||^2 = ||\sum_{n=N+1}^{M} \langle f, \phi_n \rangle \phi_n||^2 = \sum_{n=N+1}^{M} |\langle f, \phi_n \rangle|^2 \leq \sum_{n=N+1}^{\infty} |\langle f, \phi_n \rangle|^2 \to 0 \text{ as } N \to \infty.$$

Hence, by definition of Cauchy sequence (which one really should know at this point!), $\{g_N\}_{N\geq 1}$ is a Cauchy sequence in our Hilbert space. By definition of Hilbert space, every Hilbert space is complete. Thus every Cauchy sequence converges to a unique limit. Let us now call the limit of our Cauchy sequence, which is by definition,

$$\lim_{N\to\infty} g_N = \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n = \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n = g.$$

We will now show that $f - g$ satisfies

$$\langle f - g, \phi_n \rangle = 0 \forall n \in \mathbb{N}.$$

Then, because we are assuming (1) holds, this implies that $f - g = 0$, ergo $f = g$. So, we compute this inner product,

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle.$$

We insert the definition of g as the series,

$$\langle g, \phi_n \rangle = \langle \sum_{m\geq 1} \langle f, \phi_m \rangle \phi_m, \phi_n \rangle = \sum_{m\geq 1} \langle f, \phi_m \rangle \langle \phi_m, \phi_n \rangle = \langle f, \phi_n \rangle.$$

Above, we have used in the second equality the linearity of the inner product and the continuity of the inner product. In the third equality, we have used that $\langle \phi_m, \phi_n \rangle$ is 0 if $m \neq n$, and is 1 if $m = n$. Hence, only the term with $m = n$ survives in the sum. Thus,

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle f, \phi_n \rangle = 0, \quad \forall n \in \mathbb{N}.$$

By (1), this shows that $f - g = 0 \implies f = g$.

Next, we shall assume that (2) holds, and we shall use this to demonstrate (3). Well, note that

$$f = \lim_{N\to\infty} g_N \implies ||f - g_N||^2 \to 0, \quad \text{as } N \to \infty.$$

Then, by the triangle inequality,

$$||f||^2 = ||f - g_N + g_N||^2 \leq ||f - g_N||^2 + ||g_N||^2 = ||f - g_N||^2 + \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \leq |f - g_N||^2 + \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2.$$

On the other hand, by Bessel's Inequality,

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq ||f||^2.$$

So, we have a little sandwich, en smörgås, if you will, with $||f||^2$ right in the middle of our sandwich,

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq ||f||^2 \leq ||f - g_N||^2 + \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2.$$

Letting $N \to \infty$ on the right side, the term $||f - g_N|| \to 0$, and so we indeed have

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq ||f||^2 \leq \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2.$$

This of course means that all three terms are equal, because the terms all the way on the left and right side are the same!

Finally, we assume (3) holds and use it to show that (1) must also hold. This is pleasantly straightforward. We assume that for some f in our Hilbert space, $\langle f, \phi_n \rangle = 0$ for all n. Using (3), we compute

$$||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 = \sum_{n\in N} 0 = 0.$$

The only element in a Hilbert space with norm equal to zero is the 0 element. Thus $f = 0$. $\square$

8. The Best Approximation Theorem

This is another fun and cozy Hilbert room theory item.

Theorem 8.1. *Låt* $\{\phi_n\}_{n\in\mathbb{N}}$ *vara en ortonormal mängd i ett Hilbert-rum,* H*. Om* $f \in H$*,*

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \le ||f - \sum_{n\in\mathbb{N}} c_n\phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

och $=$ *gäller* $\iff c_n = \langle f, \phi_n\rangle$ *gäller* $\forall n \in \mathbb{N}$.

Proof. We make a few definitions: let

$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$

and

$$\varphi := \sum c_n\phi_n.$$

Then we compute

$$||f-\varphi||^2 = ||f-g+g-\varphi||^2 = ||f-g||^2 + ||g-\varphi||^2 + 2\Re\langle f-g, g-\varphi\rangle.$$

I claim that

$$\langle f-g, g-\varphi\rangle = 0.$$

Just write it out (stay calm and carry on):

$$\langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$

$$= \sum \overline{\widehat{f_n}}\langle f, \phi_n\rangle - \sum \overline{c_n}\langle f, \phi_n\rangle - \sum \widehat{f_n}\langle \phi_n, \sum \widehat{f_m}\phi_m\rangle + \sum \widehat{f_n}\langle \phi_n, \sum c_m\phi_m\rangle$$

$$= \sum |\widehat{f_n}|^2 - \sum \overline{c_n}\widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n}\overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have

$$||f-\varphi||^2 = ||f-g||^2 + ||g-\varphi||^2 \ge ||f-g||^2,$$

with equality iff

$$||g-\varphi||^2 = 0.$$

Let us now write out what this norm is, using the definitions of g and φ. By their definitions,

$$g - \varphi = \sum (\widehat{f_n} - c_n)\phi_n.$$

By the Pythagorean theorem, due to the fact that the ϕ_n are an orthonormal set, and hence multiplying them by the scalars, $\widehat{f_n} - c_n$, they remain orthogonal, we have

$$||g-\varphi||^2 = \sum \left|\widehat{f_n} - c_n\right|^2.$$

This is a sum of non-negative terms. Hence, the sum is only zero if all of the terms in the sum are zero. The terms in the sum are all zero iff

$$\left|\widehat{f_n} - c_n\right| = 0 \forall n \iff c_n = \widehat{f_n} \forall n \in \mathbb{N}.$$

□

9. The Magical Theorem about SLPs

This is a rather nice, follow-your-nose, theory problem. Of course, the really amazing and magical part of this theorem is the third statement, which is one of the gems of functional analysis. We shall not include that third statement here, however, because its proof is beyond the scope of this humble course.

Theorem 9.1. *Låt* f *och* g *vara egenfunktioner till ett regulärt SLP i intervallet* $[a, b]$ *med* $w \equiv 1$*. Låt* λ *vara egenvärden till* f *och* μ *vara dess till* g*. Sedan gäller:*

(1) $\lambda \in \mathbb{R}$ *och* $\mu \in \mathbb{R}$;

(2) Om $\lambda \ne \mu$*, gäller:*

$$\int_a^b f(x)\overline{g(x)}dx = 0.$$

Proof. By definition we have $Lf + \lambda f = 0$. Moreover, L is self-adjoint, so we have

$$\langle Lf, f\rangle = \langle f, Lf\rangle.$$

By definition,

$$\langle Lf, f\rangle = \int_a^b L(f)(x)\overline{f(x)}dx.$$

Thus, we have

$$-\lambda \int_a^b |f(x)|^2 dx = -\overline{\lambda}\int_a^b |f(x)|^2 dx \iff \lambda = \overline{\lambda}.$$

The last statement holds because

$$\int_a^b |f(x)|^2 dx = ||f||_{L^2} = 0 \iff f \equiv 0,$$

and by definition of f as an eigenfunction, $f \not\equiv 0$. Same proof holds for μ.

For the second part, we use basically the same argument based on self-adjointness:

$$\langle Lf, g\rangle = \langle f, Lg\rangle.$$

By assumption

$$\langle Lf, g\rangle = -\lambda\langle f, g\rangle = \langle f, Lg\rangle = \langle f, -\mu g\rangle = -\overline{\mu}\langle f, g\rangle = -\mu\langle f, g\rangle.$$

Thus, if $\langle f, g\rangle \neq 0$, this forces $\lambda = \mu$, which is false. Hence, the only viable option is that $\langle f, g\rangle = 0$. By definition,

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

□

10. The generating function for the Bessel functions

This is a lovely, follow your nose and use the definitions type of proof.

Theorem 10.1. *For all x and for all $z \neq 0$, the Bessel functions, J_n satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}.$$

Proof. We begin by writing out the familiar Taylor series expansion for the exponential functions

$$e^{xz/2} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

These converge beautifully, absolutely and uniformly for z in compact subsets of $\mathbb{C} \setminus \{0\}$. So, since we presume that $z \neq 0$, we can multiply these series and fool around with them to try to make the Bessel functions pop out... Thus, we write

$$e^{xz/2}e^{-x/(2z)} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k\geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}. \tag{10.1}$$

Here is where the one and only clever idea enters into this proof, but it's rather straightforward to come up with it. We would like a sum with $n = -\infty$ to ∞. So we look around into the above expression on the right, hunting for something which ranges from $-\infty$ to ∞. The only part which does this is $j - k$, because each of j and k range over 0 to ∞. Thus, we keep k as it is, and we let $n = j - k$. Then $j + k = n + 2k$, and $j = n + k$. However, now, we have $j! = (n + k)!$, but this is problematic if $n + k < 0$. There were no negative factorials in our original expression! So, to remedy this, we use the equivalent definition via the Gamma function,

$$j! = \Gamma(j + 1), \quad k! = \Gamma(k + 1).$$

Moreover, we observe that in (10.1), $j!$ and $k!$ are for j and k non-negative. We also observe that

$$\frac{1}{\Gamma(m)} = 0, \quad m \in \mathbb{Z}, \quad m \leq 0.$$

Hence, we can write

$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n+k+1)k!}.$$

This is because for all the terms with $n+k+1 \leq 0$, which would correspond to $(n+k)!$ with $n+k<0$, those terms ought not to be there, but indeed, the $\frac{1}{\Gamma(n+k+1)}$ causes those terms to vanish!

Now, by definition,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(k+n+1)}.$$

Hence, we have indeed see that

$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} J_n(x) z^n.$$

□

11. Orthogonality of the Hermite polynomials

This is a fun application of integration by parts many times.

Theorem 11.1. *The Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Recall here that*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

and so the statement is that

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = 0, \quad n \neq m.$$

Proof. We are showing that the weighted inner product of H_n and H_m vanishes if $n \neq m$. Hence, we may assume without loss of generality that $n > m$. Due to the fact that H_n begin with $n = 0$, this means that we must have $m \geq 0$ and $n > m$ so $n \geq 1$. Next, we insert the definition of H_n into the inner product, so we look at

$$\int_{\mathbb{R}} (-1)^n e^{x^2} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x) e^{-x^2} dx = \int_{\mathbb{R}} (-1)^n \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x) dx$$
$$= (-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x) dx.$$

Let us do integration by parts one time, since we know that $n \geq 1$. Then, we have

$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x) dx = (-1)^n \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m(x) \Big|_{x=-\infty}^{\infty}$$
$$+ (-1)^{n+1} \int_{\mathbb{R}} \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m'(x) dx.$$

The first, second, and higher order derivatives of e^{-x^2} are all of the form

$$\frac{d^n}{dx^n} e^{-x^2} = p_n(x) e^{-x^2},$$

where $p_n(x)$ is a polynomial. This follows from the chain rule. If you really want to, you can prove this by induction, but you do not need to do that on the exam. For the sake of completeness, however, I'll just go ahead and prove it. For the base case, $n = 0$, we haven't taken any derivatives, so $p_0(x) = 1$, the constant polynomial of order 0. For the first derivative, $(e^{-x^2})' = -2xe^{-x^2}$, so $p_1(x) = -2x$. Proceeding by induction, assuming $\frac{d^n}{dx^n} e^{-x^2} = p_n(x)e^{-x^2}$, then

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = \frac{d}{dx}\left(p_n(x)e^{-x^2}\right) = p_n'(x)e^{-x^2} - 2xp_n(x)e^{-x^2} = \left(p_n'(x) - 2xp_n(x)\right)e^{-x^2}.$$

The derivative of a polynomial is a polynomial, hence we have $p_{n+1}(x) = p_n'(x) - 2xp_n(x)$ is also a polynomial which proves this small fact.

Thus,

$$(-1)^n \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m(x)\Big|_{x=-\infty}^{\infty} = (-1)^n p_{n-1}(x)e^{-x^2} H_m(x)\Big|_{x=-\infty}^{\infty} = 0,$$

due to the fact that $e^{-x^2} \to 0$ as $x \to \pm\infty$ much, much faster than any polynomial tends to $\pm\infty$ as $x \to \pm\infty$. It's like Godzilla, e^{-x^2}, versus an ant, the polynomial part. Godzilla wins.

So, we have

$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x)dx = (-1)^{n+1} \int_{\mathbb{R}} \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m'(x)dx.$$

We can repeat this using the same argument, until we run out of derivatives. We've got n derivatives, so we repeat this argument n times, arriving at

$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x)dx = (-1)^{n+n} \int_{\mathbb{R}} e^{-x^2} \left(\frac{d^n}{dx^n} H_m(x)\right) dx.$$

Now, we just need to pause and think for a moment. H_m is a polynomial of degree $m < n$. If you differentiate a polynomial of degree m more than m times, you end up with nothing! Zero! So, we actually know that, because $n > m$,

$$\left(\frac{d^n}{dx^n} H_m(x)\right) = 0.$$

Hence,

$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x)dx = (-1)^{n+n} \int_{\mathbb{R}} e^{-x^2} \left(\frac{d^n}{dx^n} H_m(x)\right) dx = (-1)^{2n} \int_{\mathbb{R}} e^{-x^2} 0dx = 0.$$

□

12. The generating function for the Hermite polynomials

This is similar to the analogous result for the Bessel functions, but with a bit of a twist.

Theorem 12.1. *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

satisfy

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Proof. The key idea with which to begin is to consider instead

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2}.$$

We consider the Taylor series expansion of this guy, with respect to z, viewing x as a parameter. By definition, the Taylor series expansion for

$$e^{-(x-z)^2} = \sum_{n \geq 0} a_n z^n,$$

where

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} e^{-(x-z)^2}, \quad \text{evaluated at } z = 0.$$

To compute these coefficients, we use the chain rule, introducing a new variable $u = x - z$. Then,

$$\frac{d}{dz} e^{-(x-z)^2} = -\frac{d}{du} e^{-u^2},$$

and more generally, each time we differentiate, we get a -1 popping out, so

$$\frac{d^n}{dz^n}e^{-(x-z)^2} = (-1)^n \frac{d^n}{du^n}e^{-u^2},$$

Hence, evaluating with $z = 0$, we have

$$a_n = \frac{1}{n!}(-1)^n \frac{d^n}{du^n}e^{-u^2}, \quad \text{evaluated at } u = x.$$

The reason it's evaluated at $u = x$ is because in our original expression we're expanding in a Taylor series around $z = 0$ and $z = 0 \iff u = x$ since $u = x - z$. Now, of course, we have

$$\frac{d^n}{du^n}e^{-u^2}, \quad \text{evaluated at } u = x = \frac{d^n}{dx^n}e^{-x^2}.$$

Hence, we have the Taylor series expansion

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!}(-1)^n \frac{d^n}{dx^n}e^{-x^2}.$$

Now, we multiply both sides by e^{x^2} to obtain

$$e^{2xz-z^2} = e^{x^2} \sum_{n\geq 0} \frac{z^n}{n!}(-1)^n \frac{d^n}{dx^n}e^{-x^2}.$$

We can bring e^{x^2} inside because everything converges beautifully. Then, we have

$$e^{2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!}e^{x^2}(-1)^n \frac{d^n}{dx^n}e^{-x^2}.$$

Voilà! The definition of the Hermite polynomials is staring us straight in the face! Hence, we have computed

$$e^{2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!}H_n(x).$$

□