**Fourieranalys MVE030 och Fourier Metoder MVE290 22.augusti.2017**
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. (10 p) Låt $f$ vara en $2\pi$ periodisk funktion. Antar att $f$ är styvvis kontuerlig (piecewise continuous) och att $\forall x \in \mathbb{R}$, dess höger och vänster gränsvärde existerar:

$$\lim_{y \to x^+} f(y) = f(x_+) \in \mathbb{R}, \quad \lim_{y \to x^-} f(y) = f(x_-) \in \mathbb{R}.$$

Låt

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Bevisa att gäller:

$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} \left( f(x_-) + f(x_+) \right), \quad \forall x \in \mathbb{R}.$$

Solution is in the theory-proof compendium!

2. (10 p) Definerar Fourier transformen och ger dess Inversion-Formel.

Solution is in the theory-proof compendium!

3. (10 p) Beräkna:

$$\sum_{n=0}^{\infty} \frac{1}{4 + n^2}.$$

(Hint: Utveckla $e^{2x}$ i Fourier-series i intervallet $(-\pi, \pi)$).

We follow the hint. To do that, we compute the Fourier coefficients:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2x} e^{-inx} dx = \left. \frac{e^{2x - inx}}{2\pi(2 - in)} \right|_{x=-\pi}^{\pi} = \frac{e^{2\pi - in\pi} - e^{-2\pi + in\pi}}{2\pi(2 - in)}$$

$$= (-1)^n \frac{\sinh(2\pi)}{\pi(2 - in)}.$$

Above, we use the fact that $e^{\pm in\pi} = (-1)^n$ together with basic rules for exponentials, like $e^{a+b} = e^a e^b$, and the definition of sinh.

So, now we know that

$$e^{2x} = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad x \in (-\pi, \pi).$$

What happens for $x = \pi$ or $x = -\pi$? The series on the right does NOT converge to the function on the left!!!!! Remember Theorem 2.1! Even easier on this particular exam is THEORY QUESTION #1. It tells you (på svenska även!) what happens! When we do a Fourier expansion, we extend $e^{2x}$ from the interval $(-\pi, \pi)$ to $\mathbb{R}$ as a $2\pi$ periodic function. Doing this, the function jumps at odd-integer multiples of $\pi$. The Fourier series converges to the average of this "jump" at these points, so

$$\frac{e^{2\pi} + e^{-2\pi}}{2} = \sum_{n \in \mathbb{Z}} c_n e^{in\pi} = \sum_{n \in \mathbb{Z}} c_n (-1)^n = \sum_{n \in \mathbb{Z}} \frac{\sinh(2\pi)}{\pi(2 - in)}.$$

The left side is none other than $\cosh(2\pi)$ so we bring it together with its buddy sinh,

$$\pi \coth(2\pi) = \sum_{n \in \mathbb{Z}} \frac{1}{2 - in}.$$

Next, we take away the $n = 0$ term and pair up the $\pm n$ terms, so that

$$\pi \coth(2\pi) = \frac{1}{2} + \sum_{n \geq 1} \frac{1}{2 - in} + \frac{1}{2 + in} = \frac{1}{2} + \sum_{n \geq 1} \frac{4}{4 + n^2}.$$

Re-arranging, we have

$$\sum_{n \geq 1} \frac{1}{4 + n^2} = \frac{\pi \coth(2\pi) - 2}{4}.$$

4. (10 p) Hitta siffrorna $a_0$, $a_1$, och $a_2 \in \mathbb{C}$ som minimerar

$$\int_0^\pi |\sin(x) - a_0 - a_1 \cos(x) - a_2 \cos(2x)|^2 dx.$$

This is just expanding the sine in terms of a cosine basis on $L^2(0, \pi)$. You can probably find some stuff in $\beta$, or you can just do it by hand. The first three basis vectors here are constant multiples of $\cos(kx)$ for $k = 0, 1, 2$. These are already orthogonal, because

$$\int_0^\pi \cos(jx) \cos(kx) dx = 0 \text{ if } k \neq j.$$

So, they just need to get normalized. Thus, we compute the $L^2$ norm (squared)

$$\int_0^\pi \cos^2(kx)dx = \pi, \quad k = 0; \quad \text{or} \quad \frac{\pi}{2} \text{ for } k = 1, 2.$$

The trick to computing the integral for $k = 1, 2$ is to use the double angle formula for the cosine,

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1,$$

where we use the identity $\cos^2 + \sin^2 = 1$. Now, we have our basis vectors:

$$\frac{1}{\sqrt{\pi}}, \quad \frac{\cos(kx)\sqrt{2}}{\sqrt{\pi}}, \quad k = 1, 2.$$

Next, we compute the coefficients by computing the inner product of $\sin(x)$ with the basis vectors. It suffices for this purpose to compute:

$$\frac{1}{\sqrt{\pi}} \int_0^\pi \sin(x)dx = \frac{1}{\sqrt{\pi}}(-\cos(\pi) + \cos(0)) = \frac{2}{\sqrt{\pi}}.$$

$$\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi \sin(x)\cos(x)dx = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi \frac{1}{2}\sin(2x)dx = 0.$$

$$\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi \sin(x)\cos(2x)dx = \frac{\sqrt{2}}{\sqrt{\pi}} \left( \sin(x)\sin(2x)/2|_0^\pi - \int_0^\pi \cos(x)\frac{\sin(2x)}{2}dx \right)$$

$$= -\frac{\sqrt{2}}{\sqrt{\pi}} \left( \int_0^\pi \cos^2(x)\sin(x)dx \right)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left( \frac{\cos^3(x)}{3}\bigg|_0^\pi \right) = -\frac{2\sqrt{2}}{3\sqrt{\pi}}.$$

Hence, the best approximation of $\sin(x)$ in terms of this basis is

$$\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}} - \frac{2\sqrt{2}}{3\sqrt{\pi}}\frac{\cos(2x)\sqrt{2}}{\sqrt{\pi}} = \frac{2}{\pi} - \frac{4}{3\pi}\cos(2x).$$

The siffror we seek are therefore

$$a_0 = \frac{2}{\pi}, \quad a_1 = 0, \quad a_2 = -\frac{4}{3\pi}.$$

5. (10 p) Lös problemet:

$$u_t - u_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x,0) = e^{-x^2}$$

There's nothing like the IVP for the heat equation. We use the heat kernel (Schwartz integral kernel of the fundamental solution to the heat equation - you can learn more about Schwartz integral kernels in the future :-)

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} e^{-y^2} dy.$$

For extra fun: compute this! It isn't too bad...

6. (10 p) Beräkna

$$\int_0^\infty \frac{\sin(x)}{xe^x} dx.$$

We just need to put on our Plancharel/Parseval (I always forget which is which so just lump them together) glasses. We know that

$$\int_{\mathbb{R}} f(x)g(x)dx = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x)\hat{g}(x)dx,$$

as long as the two functions are real valued. If they're complex valued, we gotta include some complex conjugation up in there.

Well, what we've got is not an integral over $\mathbb{R}$ dagnammit. That is rather annoying. However, we can modify the integral to get an integral over $\mathbb{R}$ with a few observations. The function $\sin(x)/x$ is even. We have the product of that with $e^{-x}$. We can *extend* $e^{-x}$ to *be* an even function, using $e^{-|x|}$. So, in this way

$$\int_0^\infty \frac{\sin(x)}{xe^x} dx = \frac{1}{2} \int_{\mathbb{R}} \frac{\sin(x)}{x} e^{-|x|} dx.$$

So, while I don't really fancy doing the above integral, using the Parseval/Plancharel trick, we can replace those functions by the Fourier transforms:

$$\frac{1}{2} \int_{\mathbb{R}} \frac{\sin(x)}{x} e^{-|x|} dx = \frac{1}{4\pi} \int_{\mathbb{R}} \pi \chi_{(-1,1)}(x) \frac{2}{x^2+1} dx = \frac{1}{2} \int_{-1}^{1} \frac{1}{x^2+1} dx$$

$$= \frac{1}{2} \arctan(x)|_{-1}^{1} = \frac{1}{2} \left( \frac{\pi}{4} - -\frac{\pi}{4} \right) = \frac{\pi}{4}.$$

How cute.

7. (10 p) Lös problemet:

$$u_{xx} + u_{yy} = -20u, \quad 0 < x < 1, \quad 0 < y < 1,$$

$$\begin{aligned} u(0,y) &= u(1,y) = 0, \\ u(x,0) &= 0, \\ u(x,1) &= x^2 - x. \end{aligned}$$

We begin by separating variables, writing $u = XY$. Then, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = -20.$$

This means that $X''/X$ and $Y''/Y$ must both be constant, and we write

$$\frac{X''}{X} = -20 - \frac{Y''}{Y} = \mu.$$

The BCs for $X$ are nicer, so we start with $X$. We have

$$X'' = \mu X, \quad X(0) = X(1) = 0.$$

You can show that the only $\mu$ which have a non-trivial solution $X$ are $\mu < 0$, specifically,

$$X = X_n = \sin(n\pi x), \quad \mu = \mu_n = -n^2\pi^2, \quad n \in \mathbb{N}, n \geq 1,$$

up to a constant factor. Then, this also specifies the partner solution, because we know that $Y$ satisfies

$$\frac{Y_n''}{Y_n} = -\frac{X''}{X} - 20 = n^2\pi^2 - 20 = \lambda_n.$$

For $n = 1$, we note that $\lambda_1 < 0$. Thus, we have $Y_1$ is a linear combination of $\sin(\sqrt{|\lambda_1|}y)$ and $\cos(\sqrt{|\lambda_1|}y)$. For $n \geq 2$, $\lambda_n > 0$, so there $Y_n$ is a linear combination of $\sinh(\sqrt{\lambda_n}y)$ and $\cosh(\sqrt{\lambda_n}y)$. To figure out the constant factors, we use the BCs. We need $Y_n(0) = 0$ for all $n$. Thus,

$$Y_1(y) = \sin(\sqrt{|\lambda_1|}y), \quad Y_n(y) = \sinh(\sqrt{\lambda_n}y), \quad n \geq 2,$$

up to multiplication by a constant factor. Our full solution is then given by summing

$$u(x, y) = \sum_{n \geq 1} a_n \sin(n\pi x) Y_n(y).$$

We need

$$u(x, 1) = \sum_{n \geq 1} a_n \sin(n\pi x) Y_n(1) = x^2 - x.$$

Hence, we need to expand the function $x^2 - x$ in terms of the $L^2(0, 1)$ OB (not yet normalized) $\{\sin(n\pi x)\}$. We compute the $L^2$ norms of the sines to be $1/\sqrt{2}$. Hence, the Fourier coefficients shall be

$$c_n = \frac{1}{2} \int_0^1 (x^2 - x) \sin(n\pi x) dx.$$

Then, the coefficients, $a_n$ are given by

$$a_n = \frac{c_n}{Y_n(1)}.$$

We note that $Y_1(1) = \sin(\sqrt{20 - \pi^2}) \neq 0$, and that sinh has no zeros on the real line. So, phew, we aren't dividing by zero.

8. (10 p) Lös problemet:

$$u_t - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$\begin{aligned} u(0, t) &= t + 1, \\ u(1, t) &= 0, \\ u(x, 0) &= 1 - x. \end{aligned}$$

Staring at those weird BCs, we see that

$$(t + 1)(1 - x) = (t + 1) \text{ at } x = 0,$$

and

$$(t + 1)(1 - x) = 0 \text{ at } x = 1,$$

and

$$(t + 1)(1 - x) = (1 - x) \text{ at } t = 0.$$

What happens if we hit $(t+1)(1-x)$ with the heat equation? We get

$$(\partial_t - \partial_x^2)(t+1)(1-x) = 1-x.$$

So, we look for a steady state solution to:

$$-f''(x) = x-1 \implies f(x) = -\frac{x^3}{6} + \frac{x^2}{2} + ax + b.$$

Now, because $(t+1)(1-x)$ takes care of the BCs, we want $f$ to vanish at the boundaries. So, we want

$$f(0) = f(1) = 0 \implies b = 0 \text{ and } a = -\frac{1}{3}.$$

However, now the function $f$ is going to screw up the IC, so we gotta fix it by finding $v$ which satisfies

$$v_t - v_{xx} = 0, \quad 0 < x < 1, \quad t > 0,$$

$$v(0,t) = v(1,t) = 0,$$

$$v(x,0) = -f(x),$$

and our full solution will be

$$u(x,t) = (t+1)(1-x) + f(x) + v(x,t).$$

This is just an IVP for the standard heat equation! We can solve it using separation of variables and a Fourier series. When we do that, we get

$$T'X - TX'' = 0 \implies X'' = \text{ constant } X,$$

with BCs

$$X(0) = X(1) = 0.$$

Hence,

$$X_n(x) = \sin(n\pi x) \text{ up to constant factor.}$$

We then also get

$$T_n(t) = e^{-n^2\pi^2 t},$$

up to constant factor. Our full solution is

$$v(x,t) = \sum_{n\geq 1} a_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

To get the constants, we use the IC which says

$$v(x,0) = \sum_{n\geq 1} a_n \sin(n\pi x) = -f(x) = \frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3}.$$

The coefficients are therefore given by

$$a_n = 2\int_0^1 (\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3}) \sin(n\pi x) dx.$$

The 2 in front comes from the fact that the $L^2$ norm of the basis vectors $\sin(n\pi x)$ is $1/\sqrt{2}$.