FOURIER ANALYSIS & METHODS

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2018.01.15

According to Gerry, Fourier Analysis is "A collection of related techniques for solving the most important partial differential equations of physics (and chemistry)." For example, we're going to be solving

- Δ Laplace equations (related to computing energy of quantum particles)
- $\Box$ wave equations (describes the propagation of waves, hence also of light and electromagnetic waves)
- heat equation (describes the propagation of heat, is the quintessential diffusion equation)

A general feature of PDEs is that they are HARD. We can only solve them using the cleverest of clever tricks. So, in this course, we're going to fill your bag with tricks. Or if you prefer to think about tools and a toolbox, we're going to fill your toolboxes.

1.1. **Technique 0: Separation of variables.** If you come to the (obligatory for Kf, option for TM and F) extra three lectures, you'll learn how to classify every PDE on the planet. For the great majority of these, we have no hope to solve then analytically (that is, to write down some mathematical formula for the answer). However, for a special few, we can solve them. The first tricky way to solve a PDE, that is an equation for an unknown function which depends on several variables, all jumbled up together, is to separate those jumbled up variables. If we can do that, then we have a ray of hope of turning the PDE into one or more ODEs. ODEs are called ordinary for a reason. They're simpler than PDEs.

So, to introduce the technique of separation of variables, let's think about a really down-to-earth example. A vibrating string. (Could be on a guitar, piano, violin, or whatever instrument you prefer). Then, of course, the ends of the string are held fixed, so they're not moving. You know this if you play guitar. One end is fixed at the bottom, and you slide your hand to various places on the string, to hold it down, to get the notes you want. We're going to see something rather interesting mathematically about how this all works. Let's mathematicize the string, by identifying it with the interval $[0, \ell] \subset \mathbb{R}$. The string length is ℓ. Let's define

$$u(x,t) := \text{ the height of the string at the point } x \in [0,\ell] \text{ at time } t \in [0,\infty[.$$

Then, let's just define the sitting-still height to be height 0. So, the ends not moving up or down mean

$$u(0,t) = u(\ell,t) = 0 \quad \forall t.$$

A positive height means above the sitting-still height, whereas a negative height means under the sitting-still height. The wave equation (I'm not going to derive it, but maybe you clever physics students can do that?)

$$u_{xx} = c^2 u_{tt}.$$

The constant c depends basically on how fast the string vibrates. Now, because we're in a math class, we're going to use funky time units and just assume $c = 1$. The reason we can do that is if we defined time units $\tau = ct$, then $u_{\tau\tau} = c^2 u_{tt}$. So, using the funky time units τ (which are just the old t time units, whatever they were, scaled by c), the wave equation becomes

$$u_{xx} = u_{\tau\tau}.$$

To keep life simple though, we're going to just use t and assume $c = 1$. No generality is lost by doing this for the aforementioned (time units) reason. What we've got is a PDE, because we got both x derivatives and t derivatives, and u depends on both x and t. Eek. The separation of variables idea is to *assume* we can write

$$u(x,t) = f(x)g(t),$$

that is a product of two functions, each of which depends only on *one* variable. (Whether this assumption is kosher remains to be determined...) Now, assuming that u is of this form, we write the PDE

$$u_{xx} = u_{tt} \iff f''(x)g(t) = f(x)g''(t).$$

Doesn't look much better yet, but hang on there. Divide both sides by $f(x)g(t)$. We get

$$\frac{f''}{f}(x) = \frac{g''}{g}(t).$$

Stop. Think. The left side depends only on x, whereas the right side depends only on t. Hence, they both must be constant. We've got more information on x than we do on t, because we know that the ends are still. This means that

$$f(0) = f(\ell) = 0.$$

So, the equation for just f is

$$\frac{f''}{f}(x) = \text{ constant },$$
$$f(0) = f(\ell) = 0.$$

Let's give the constant a name. Call it λ. Then write

$$f''(x) = \lambda f(x), \quad f(0) = f(\ell) = 0.$$

Well, we can solve this. There are three cases to consider:

$\lambda = 0$ This means $f''(x) = 0$. Integrating both sides once gives $f'(x) =$ constant $= m$. Integrating a second time gives $f(x) = mx + b$. Requiring $f(0) = f(\ell) = 0$, well, the first makes $b = 0$, and the second makes $m = 0$. So, the solution is $f(x) \equiv 0$. The 0 solution. The waveless wave. Not too interesting.

$\lambda > 0$ The solution here will be of the form

$$f(x) = ae^{\sqrt{\lambda}x} + be^{\sqrt{\lambda}x}.$$

Similarly, you can compute that to get $f(0) = f(\ell) = 0$, we'll need $a = b = 0$. It's the 0 solution again. The waveless wave. No fun there.

$\lambda < 0$ Finally, we have solution of the form

$$a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To make $f(0) = 0$, we need $a = 0$. Uh oh... are we going to get that stupid 0 solution again? Well, let's see what we need to make $f(\ell) = 0$. For that we just need

$$b\sin(\sqrt{|\lambda|}\ell) = 0.$$

That will be true if

$$|\lambda| = \frac{k^2\pi^2}{\ell^2}, \quad k \in \mathbb{Z}.$$

Super! We still don't know what b ought to be, but at least we've found all the possible f's, up to constant factors.

Just to clarify the fact that we've now found *all* solutions, we recall here a theorem from your multivariable calculus class.

Theorem 1 (Old Multivariable Calculus Theorem)**.** *A basis of solutions to the second order linear homogeneous ODE, with $a \neq 0$,*

$$au'' + bu' + cu = 0$$

is one of the following three mutually exclusive sets:

(1) $\{e^{r_1x}, e^{r_2x}\}$ if $b^2 > 4ac$ in which case

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(2) $\{e^{rx}, xe^{rx}\}$ if $b^2 = 4ac$, in which case $r = -\frac{b}{2a}$.

(3) $\{\sin(\Im rx)e^{\Re rx}, \cos(\Im rx)e^{\Re rx}\}$ if $b^2 < 4ac$, in which case $r = -\frac{b}{2a} + \frac{i}{2a}\sqrt{4ac - b^2}$.

Our equation had $b = 0$, $a = 1$, and $c = -\lambda$. So, it is a good **exercise** to go through the paces of putting the equation we just solved into the language of the theorem, and verifying that everything checks out.

So, the solutions we've found are:

$$f_k(x) = \sin\left(\frac{k\pi x}{\ell}\right), \quad \lambda_k = -\frac{k^2\pi^2}{\ell^2}.$$

Now, let's find the friends of f, the time functions. When we've got f_k, then

$$\frac{f_k''}{f_k} = \lambda_k = -\frac{k^2\pi^2}{\ell^2} = \frac{g_k''}{g_k}.$$

Hence, we know (up to constant factors)

$$g_k(t) = a_k\cos\left(\frac{k\pi t}{\ell}\right) + b_k\sin\left(\frac{k\pi t}{\ell}\right).$$

Let us pause to think about what this means. The physics students may recognize that the numbers

$$\{|\lambda_k|\}_{k\geq 1}$$

are the resonant frequencies of the string. Basically, they determine how it sounds. The number $|\lambda_1|$ is the fundamental tone of the string. The higher $|\lambda_k|$ for $k \geq 2$ are harmonics. It is interesting to note that they are all square-integer multiples of λ_1. Here's a question: if you can "hear" the value of $|\lambda_1|$, then can you tell me how long the string is? Well, yes, cause

$$|\lambda_1| = \frac{1}{\ell^2}, \implies \ell = \frac{1}{\sqrt{|\lambda_1|}}.$$

So, you can hear the length of a string. A couple of famous unsolved math problems: can one hear the shape of a convex drum? Can one hear the shape of a smoothly bounded drum? We can talk about these problems if you're interested.

So, now that we've got all these solutions, what should we do with them? Good question...

1.2. **Superposition principle and linearity.** Superposition basically means adding up a bunch of solutions. You can think of it like adding up a bunch of solutions to get a super solution!

Definition 2. A second order linear PDE for an unknown function u of n variables is an equation for u and its mixed partial derivatives up to order two of the form

$$L(u) = f,$$

where f is a given function, and there are known functions $a(x)$, $b_i(x)$, $c_{ij}(x)$ for $x \in \mathbb{R}^n$ such that

$$L(u) = a(x)u(x) + \sum_{i=1}^n b_i(x)u_{x_i}(x) + \sum_{i,j=1}^n c_{ij}(x)u_{ij}(x).$$

In this context, L is called a *second order linear partial differential operator.*

The reason it's called linear is because you can check that for two functions u and v,

$$L(u+v) = L(u) + L(v).$$

Moreover, for any constant $c \in \mathbb{R}$, we have

$$L(cu) = cL(u).$$

Definition 3. The wave operator, $\Box$, defined for $u(x,y)$ with $(x,y) \in \mathbb{R}^2$ is

$$\Box(u) = -u_{xx} + u_{tt}.$$

Exercise 1. *Verify that the wave operator is a second order linear partial differential operator.*

We have shown that the functions

$$u_k(x,t) = f_k(x)g_k(t)$$

satisfy

$$\Box u_k = 0 \forall k.$$

Hence, if we add them up this remains true:

$$\Box(u_1 + u_2 + u_3 + \dots) = 0.$$

OBS![1] On the other hand, the equations

$$f_k'' = \lambda_k f_k \iff f_k'' - \lambda_k f_k = 0$$

do *not* add up. This is because the λ_k are different. So, it's not true that

$$f_1'' + f_2'' - (\lambda_1 + \lambda_2)(f_1 + f_2) = 0.$$

Just check it. The reason is that if we want to write it operator-style, then there's a bunch of operators

$$L_k(u) = u'' - \lambda_k u.$$

They're different operators for different k's. So, always take care when smashing solutions (i.e. superposing) together!

However, when we look at the different $u_k(x,t)$ in the *wave equation,* it's all good. So, we've still got some unanswered questions:

(1) What are the constants a_k and b_k?
(2) If we can figure out what the constants are, and then write

$$\sum_{k\geq 1} \sin\left(\frac{k\pi x}{\ell}\right)\left(a_k \cos(k\pi t/\ell) + b_k \sin(k\pi t/\ell)\right),$$

is this mess going to converge?

Let's think about what happens when you play guitar. You gotta strum it or pluck it to make a sound. So, we think of the instant when you pluck or strum the guitar, and call that time $t = 0$. Then, there is a function

$$u_0(x) = \text{ the height at the point } x \text{ on the string at time } t = 0.$$

If we just substitute rather blindly $t = 0$ into the series above, we get

$$\sum_{k\geq 1} \sin(k\pi x/\ell) a_k.$$

So, we're going to want that sum to equal $u_0(x)$. When can we do that? Is it always possible?

Finally, a note of caution. We *started* by separating variables. This lead us to the equations for the unknowns f, g, and the unknown constant λ. The assumption that the ends of the string are fixed, not moving, is called *a boundary condition.* The "place" where the action is happening is a string, identified with $[0, \ell] \subset \mathbb{R}$. That's a set. It's got a boundary. The boundary consists of the two endpoints. We saw that the condition at the boundary (immobile string) determined which values λ can have. It turned out (for super deep awesome mathematical reasons in fact) that we get a countable set of λ's, which we indexed by the natural numbers. Then, we used these λ_k to find the "friend functions" the $g_k(t)$. Finally, we saw that if we superimpose all the $u_k(x,t) = f_k(x) g_k(t)$, we still get a solution to the wave equation. It's important to note that in the end, this big

$$\sum_{k\geq 1} u_k(x,t),$$

is NOT of the form $F(x)G(t)$. So, the separation of variables technique is just a part of the whole big picture. It's a tool, but it's not in general the final picture.

[1] I love this Swedish expression. Nothing quite like it in the languages I know. Well, the closest is maybe 小心 which is also very cute.

Next time we'll play around with the heat equation and investigate the question about these a_k, b_k, and the summability of such a sequence of trig functions....