FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Would you rather fight 100 duck sized horses or 1 horse sized duck? Even a big opponent, if it's just one, is easier to manage than 100 duck sized horses all at once! A more depressing example is Russia in the second world war. Their soldiers were (generally speaking) not as well equipped and trained as the Nazis. It is a Russian saying that they defeated the Nazis by drowning them in Russian blood. So, even though the Nazis were (generally speaking) more skilled, the Russians were so numerous that they eventually drove the Nazis out of Russia. Well, this is a vastly over-simplified version of how my Russian fiancé (whose grandfather fought in WW2) describes it. I am just a mathematician, not knowing much about history. Do have some cool WW2 stories from my own grandpa though. He was a fighter pilot. Anyways, how this relates to math is that fighting too many opponents at the same time is poor strategy. Don't do it.

1.0.1. *General principle: divide and conquer.* Ideally, you want to deal with inhomogeneous parts one at a time. So, you break the problem down into pieces and try to solve the pieces: divide and conquer. Deal with each inhomogeneity one at a time. Then add them up. It is difficult to give a definitive formula that one can mindlessly use in every situation (like the formula for the solutions to $ax^2 + bx + c = 0$). The best tactic is to keep these principles and examples in mind (and at hand for reference while you are in the practicing/learning phase), and to just do lots and lots of problems. Occasionally though, especially in future "real world problems" you may come to a PDE which *has no solution.* Like that weird one on the quiz. So, if you are really struggling, consider the possibility that maybe what you're trying to do is impossible. On exams, though, this won't happen. In the real (research & applied) world though....

1.0.2. *IVP for inhomogeneous PDE with time independent inhomogeneity.* Let's consider the problem

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

$$u(-\pi,t) = u(\pi,t) = 0$$

FIGURE 1. Just for fun, here is an old photo of my grandpa in his plane. Coincidentally, he is a first generation Swede (his parents immigrated from Sweden to the USA in the early 1900s). Note that I wrote *is,* because he's going on 97.

$$u_t(x,0) = 0, \quad x \in [-\pi, \pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 5 \quad x \in [-\pi, \pi], \quad t > 0.$$

OH NO! It's not a homogeneous PDE! What do we do?!?!? Don't panic. Observe that the inhomogeneity is *independent of* t.

> **Idea:** Deal with time independent inhomogeneity in the PDE by finding a steady state solution.

We look for a function $f(x)$ which depends only on x which satisfies the boundary conditions and also satisfies the inhomogeneous PDE. Since f only depends on x, the PDE for f is

$$-f''(x) = 5.$$

This means that

$$-f'(x) = 5x + b \implies -f(x) = \frac{5x^2}{2} + bx + c \implies f(x) = -\frac{5x^2}{2} + bx + c.$$

Now, we want f to satisfy the boundary conditions. So, we want

$$-\frac{5\pi^2}{2} - b\pi + c = 0 = -\frac{5\pi^2}{2} + b\pi + c \iff b = 0.$$

There is no restriction on c, so we may as well let $c = 0$ for the sake of simplicity. So, we now have a solution to

$$-f''(x) = 5, \quad f(\pm\pi) = 0,$$

which is

$$f(x) = -\frac{5x^2}{2}.$$

If we then look for a solution to

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases} =: v(x)$$

$$u(-\pi,t) = u(\pi,t) = 0$$

$$u_t(x,0) = 0, \quad x \in [-\pi,\pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0,$$

and we add it to f, we will get

$$u(x,0) + f(x) = v(x) + f(x) \neq v(x).$$

The initial condition gets messed up because of f. So, we need to compensate for this. For that reason, we look for a solution to

$$u(x,0) = -f(x) + v(x)$$

$$u(-\pi,t) = u(\pi,t) = 0$$

$$u_t(x,0) = 0, \quad x \in [-\pi,\pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0.$$

Then, our full solution will be

$$U(x,t) = u(x,t) + f(x).$$

It will now satisfy *everything*. Here it is important to note that when we add u and f, the boundary condition still holds. So, please think about this, because in certain variations on the theme, it could possibly not be true.

Anyhow, we are now just dealing with this nice IVP for the homogeneous wave equation. We can recycle what we did on Friday. We had the heat equation there, but watch what happens when we separate variables:

$$T''(t)X(x) - X''(x)T(t) = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \lambda,$$

is a constant. So, for the X part, we have the problem:

$$X'' = \lambda X, \quad X(\pm\pi) = 0.$$

This is the SLP we solved on Friday. If you need to review how we did that, please go back to the Day9 notes and read it! I will just *skip to the good bit:*[1]

$$X_n(x) = a_n \cos(\sqrt{|\lambda_n|}x) + b_n \sin(\sqrt{|\lambda_n|}x),$$

$$a_n = \begin{cases} 0 & n = \text{ even} \\ \left(\int_{-\pi}^{\pi} \cos(\sqrt{|\lambda_n|}x)^2 dx\right)^{-\frac{1}{2}} & n = \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & n = \text{ odd} \\ \left(\int_{-\pi}^{\pi} \sin(\sqrt{|\lambda_n|}x)^2 dx\right)^{-\frac{1}{2}} & n = \text{ even} \end{cases}$$

with

$$\sqrt{|\lambda_n|} = \frac{n}{2}, \quad \lambda_n = -\frac{n^2}{4}.$$

The partner functions,

$$T_n(t) = \alpha_n \cos(\sqrt{|\lambda_n|}x) + \beta_n \sin(\sqrt{|\lambda_n|}x).$$

We shall determine the coefficients using the IC. First, we write

$$u(x,t) = \sum_{n\geq 1} T_n(t) X_n(x).$$

Next, we use the easier of the two ICs, which is

$$u_t(x,0) = 0.$$

So, we also compute

$$u_t(x,t) = \sum_{n\geq 1} T_n'(t) X_n(x).$$

When we plug in 0, we need to have

$$u_t(x,0) = \sum_{n\geq 1} T_n'(0) X_n(x) = 0.$$

So, to get this, we need

$$T_n'(0) = 0 \forall n.$$

By definition of the T_n,

$$T_n'(0) = \beta_n \sqrt{|\lambda_n|}.$$

So, to make this zero, since $\sqrt{|\lambda_n|} \neq 0$, we need

$$\beta_n = 0 \forall n.$$

Hence, our solution looks like

$$u(x,t) = \sum_{n\geq 1} \alpha_n \cos(\sqrt{|\lambda_n|}t) X_n(x).$$

The other IC says

$$u(x,0) = -f(x) + v(x).$$

Since $\cos(0) = 1$, we see that we need

$$-f(x) + v(x) = \sum_{n\geq 1} \alpha_n X_n(x).$$

[1]This is a really fun song by the rap duo, Rizzle Kicks.

This means that we need

$$\alpha_n = \langle -f + v, X_n\rangle = \int_{-\pi}^{\pi} \left(-f(x) + v(x)\right) X_n(x)dx.$$

It suffices to just leave α_n like this. As we observed before, our full solution is now

$$U(x,t) = u(x,t) + f(x) = -\frac{5x^2}{2} + \sum_{n\geq 1} \alpha_n \cos(\sqrt{|\lambda_n|}x)X_n(x),$$

with X_n defined as above.

1.0.3. *IVP for homogeneous PDE with non-self-adjoint BCs.* Let's say we have the problem

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 20. \end{aligned}$$

These are not self adjoint BCs. Yikes! However, we can use a similar "steady state" trick to deal with this. If the BC $u(0,t) = 20$ were instead $u(0,t) = 0$, then the BCs would be self adjoint BCs. So we want to make it so. Since the PDE is homogeneous, the

Idea: Deal with non-self adjoint BCs which are independent of time by finding a steady state solution.

So, we want a function $f(x)$ which satisfies the equation

$$-f''(x) = 0,$$

and which gives us the bad BC

$$f(0) = 20.$$

We have a nice homogeneous BC on the other side, so we don't want to mess that up, so we want

$$f'(4) = 0.$$

Then, the function

$$f(x) = -\frac{5x^2}{2} + ax + b.$$

We use the BCs to compute

$$f(0) = 20 \implies b = 20.$$

$$f'(4) = 0 \implies -\frac{5*16}{2} + 4a + 20 = 0 \implies 40 = 20 + 4a \implies 20 = 4a \implies a = 5.$$

Thus,

$$f(x) = -\frac{5x^2}{2} + 5x + 20.$$

Similar to before, if we add it to the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$u(x,0) = v(x),$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 0.$$

it's going to screw up the IC. So, instead we look for the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$u(x,0) = v(x) - f(x),$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 0.$$

We can now deal with this in the standard way. We use SV to write $u = XT$ (just a means to an end).[2] Next, we get the equation

$$\frac{T'}{T} = \frac{X''}{X} = \lambda.$$

We solve the SLP

$$X'' = \lambda X, \quad X(0) = 0 = X'(4).$$

The reason we know this is an SLP satisfying the hypotheses of the theorem is because we verify that the BC is self-adjoint. I leave this as an **exercise** for you to do. We look for solutions for the three cases of λ. I leave it as an **exercise** for you to show that $\lambda \geq 0$ has no non-zero solutions. We only get $\lambda < 0$. Then, the solution is of the form

$$a_n \cos(\sqrt{|\lambda_n|}x) + b_n \sin(\sqrt{|\lambda_n|}x).$$

The BC at 0 tells us that

$$a_n = 0.$$

The BC at 4 tells us that

$$\cos(\sqrt{|\lambda_n|}4) = 0 \implies \sqrt{|\lambda_n|}4 = \frac{2n+1}{2}\pi \implies \sqrt{|\lambda_n|} = \frac{2n+1}{8}\pi.$$

We then also get

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{64}.$$

The coefficient

$$b_n = \left(\int_0^4 \sin^2(\sqrt{|\lambda_n|}x) dx \right)^{-\frac{1}{2}}.$$

Hence,

$$X_n(x) = b_n \sin(\sqrt{|\lambda_n|}x).$$

I do believe if you compute it, you shall get $b_n = \frac{1}{\sqrt{2}}$. You may check this if you are so inclined. The partner function

$$\frac{T_n'}{T_n} = \lambda_n \implies T_n(t) = \alpha_n e^{\lambda_n t} = \alpha_n e^{-(2n+1)^2\pi^2 t/64}.$$

[2] La fin justifie les moyens by M.C. Solar is recommended listening.

We put it all together writing

$$u(x,t) = \sum_{n\geq 1} T_n(t)X_n(x).$$

To make the IC, we need

$$u(x,0) = \sum_{n\geq 1} T_n(0)X_n(x) = v(x) - f(x).$$

Since

$$T_n(0) = \alpha_n,$$

we need

$$\sum_{n\geq 1} \alpha_n X_n(x) = v(x) - f(x).$$

So we want the coefficients to be the Fourier coefficients of $v - f$, thus

$$\alpha_n = \langle v - f, X_n \rangle = \int_0^4 (v(x) - f(x))\, \overline{X_n(x)} dx.$$

Our full solution is

$$U(x,t) = u(x,t) + f(x) = -\frac{5x^2}{2} + 5x + 20 + \sum_{n\geq 1} T_n(t)X_n(x).$$

1.0.4. *IVP for inhomogeneous PDE with time* dependent *inhomogeneity.* Solve:

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$u(x,0) = v(x),$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 0.$$

Non! Sacre bleu! Tabernac![3] There's a lovely way to deal with this type of inhomogeneity. We first solve the homogeneous problem. It is quite similar to what we have just done. We get the same

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{64}, \quad b_n = \left(\int_0^4 \sin^2(\sqrt{|\lambda_n|}x) dx \right)^{-\frac{1}{2}}.$$

$$X_n(x) = b_n \sin(\sqrt{|\lambda_n|}x).$$

$$T_n(t) = \alpha_n e^{\lambda_n t}.$$

$$\alpha_n = \langle v, X_n \rangle = \int_0^4 v(x)\overline{X_n(x)} dx = \hat{v}_n.$$

Let us now call

$$w(x,t) = \sum_{n\geq 1} T_n(t)X_n(x) = \sum_{n\geq 1} \hat{v}_n e^{\lambda_n t} X_n(x).$$

So, this solves everything except the creepy tx part. We shall deal with that part by looking for a solution to

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

[3]This is how they curse in French Canada.

$$u(x,0) = 0,$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 0.$$

Idea: look for a solution of the form

$$\sum_{n\geq 1} c_n(t) X_n(x).$$

So, we keep our X_n from the homogeneous problem's SLP, and we look for different c_n. We want the function to satisfy

$$u_t - u_{xx} = tx,$$

so we put the series in the left side

$$\sum_{n\geq 1} c_n'(t) X_n(x) - c_n(t) X_n''(x) = tx.$$

We use the fact the $X_n'' = \lambda_n X_n$, so we want to solve

$$\sum_{n\geq 1} X_n(x) \left(c_n'(t) - c_n(t)\lambda_n\right) = tx.$$

Here is where we do something clever:

Idea: write out tx as a Fourier series in terms of X_n.

The t just goes along for the ride, and

$$tx = t \sum_{n\geq 1} \hat{x}_n X_n(x),$$

where

$$\hat{x}_n = \langle x, X_n \rangle = \int_0^4 x X_n(x) dx.$$

Consequently we want

$$\sum_{n\geq 1} X_n(x) \left(c_n'(t) - c_n(t)\lambda_n\right) = tx = \sum_{n\geq 1} t X_n(x) \hat{x}_n.$$

We equate the coefficients

$$\left(c_n'(t) - \lambda_n c_n(t)\right) = t\hat{x}_n.$$

This is an ODE for $c_n(t)$. We also want the IC, $c_n(0) = 0$. The solution to the homogeneous ODE $f' - \lambda_n f = 0$ are functions of the form

$$\alpha_n e^{\lambda_n t}.$$

A particular solution to the inhomogeneous ODE is the

$$-\frac{\hat{x}_n}{\lambda_n} t - \frac{\hat{x}_n}{\lambda_n^2}.$$

In order to get the condition that $c_n(0) = 0$ we take the solution to the homogeneous ODE, choose α_n correctly, and add it to the particular solution to the inhomogeneous ODE. This gives us

$$c_n(t) = -\frac{\hat{x}_n}{\lambda_n} t - \frac{\hat{x}_n}{\lambda_n^2} + \frac{\hat{x}_n}{\lambda_n^2} e^{\lambda_n t}.$$

(If you forget how to solve the ODE, you can just say in your solution that you need $c_n(t)$ to solve the ODE with the correct IC). Therefore the solution we seek is

$$u(x,t) = \sum_{n\geq 1} c_n(t) X_n(x),$$

and the full solution is

$$U(x,t) = w(x,t) + u(x,t).$$

1.1. **Variations on the theme.** This is from an old exam, so the notation may be different (like using λ_n^2 and stuff). It is good to get used to different notations, so I leave it this way for your edification. We probably are not doing these examples in lecture, but it may be good for you to practice with them, so I'm including them in the notes.

1.1.1. *Inhomogeneous PDE* and *BC.* Solve:

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$u(x,0) = 20,$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 20.$$

There are a few ways to deal with this. The straightforward way is to:

(1) Deal with the BC by finding a steady state solution for the BC (the $u(0,t) = 20$ part). That is find a function $f(x)$ to solve the homogeneous PDE, $-f''(x) = 0$ with the BCs $f'(4) = 0$, $f(0) = 20$.
(2) Next, solve the homogeneous PDE with the IC $u(x,0) = 20$. Oh wait, that's impossible. So don't do that.
(3) Try again: next, solve the inhomogeneous PDE but with the nice BCs $u(0,t) = 0 = u_x(4,t)$. To do this use SLP to find the X_n and λ_n. Then look for a series solution of the form
$$\sum_{n\geq 1} c_n(t) X_n(x),$$
where $c_n(t)$ is going to satisfy an ODE. This comes from plugging the series into the PDE and expanding the function tx on the right in a Fourier series with respect to $\{X_n\}$.
(4) Combine this with your $f(x)$ to get the full solutions.

Exercise: DO THIS. As described above. Do not read the solution below. Only after you have done this, you may read the solution below and verify that you get the same answer.

The boundary conditions and initial condition are inhomogeneous. So, we first solve the homogeneous PDE with these inhomogeneous conditions. It's pretty simple, because the constant function 20 does the job.

Next, we solve the inhomogeneous PDE but with homogeneous BC and IC, specifically, we now solve

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$u(x,0) = 0,$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 0.$$

If we add the solution to the constant, 20, then the sum will do the job. First, we think about the homogeneous PDE, which would give us

$$\frac{T'}{T} - \frac{X''}{X} = 0 \implies \frac{X''}{X} = \frac{T'}{T} = \text{ constant.}$$

We have the nice boundary conditions for X,

$$X(0) = X'(4) = 0 \implies X_n(x) = \sin((n+1/2)\pi x/4),$$
$$X_n''(x) = -\lambda_n^2 X_n(x), \quad \lambda_n = \frac{(n+1/2)\pi}{4}.$$

up to constant factor. By the SLP theory, these guys form an orthogonal basis for $L^2(0,4)$, so we can expand the function tx in terms of this basis,

$$tx = t\sum_{n\geq 0} \widehat{x_n} X_n(x),$$

where

$$\widehat{x_n} = \frac{1}{2}\int_0^4 x\sin((n+1/2)\pi x/4)dx = \frac{8(-1)^n}{(n+1/2)^2}.$$

Now, we set up the PDE for

$$u(x,t) = \sum_{n\geq 1} c_n(t) X_n(x).$$

We apply the heat operator, and we want to solve

$$\sum_{n\geq 1} c_n'(t)X_n(x) - c_n(t)X_n''(x) = tx = \sum_{n\geq 1} t\widehat{x_n}X_n(x).$$

We use the equation satisfied by X_n to change this around to

$$\sum_{n\geq 1}\left(c_n'(t) + \lambda_n^2 c_n(t)\right) X_n(x) = \sum_{n\geq 1} t\widehat{x_n}X_n(x).$$

We equate coefficients,

$$c_n'(t) + \lambda_n^2 c_n(t) = t\widehat{x_n}.$$

This is an ODE. We also have the IC, that we want $c_n(0) = 0$. A particular solution to the ODE is a linear function of t, that is

$$at + b.$$

Let's substitute a function of that type into the ODE above,

$$a + \lambda_n^2(at+b) = t\widehat{x_n}.$$

Then equating coefficients, we need that

$$a = a_n = \frac{\widehat{x_n}}{\lambda_n^2}, \quad a + \lambda_n^2 b = 0 \implies b = \frac{-a}{\lambda_n^2} = \frac{-\widehat{x_n}}{\lambda_n^4}.$$

The particular solution is then

$$\frac{\widehat{x_n}}{\lambda_n^2}\left(t - \lambda_n^{-2}\right).$$

We would like $c_n(0) = 0$. However, this is not necessarily the case above. How to remedy this dilemma? We include a solution to the homogeneous ode,

$$f' + \lambda_n^2 f = 0.$$

This is solved by constant multiples of $e^{-\lambda_n^2 t}$. So, the solution we seek is then

$$c_n(t) = \frac{\widehat{x_n}}{\lambda_n^2}\left(t - \lambda_n^{-2}\right) + b_n e^{-\lambda_n^2 t}.$$

Setting $c_n(0) = 0$, we see that the constant we seek is

$$b_n = \frac{\widehat{x_n}}{\lambda_n^4}.$$

Thus

$$c_n(t) = \frac{\widehat{x_n}}{\lambda_n^2}\left(t - \frac{1}{\lambda_n^2} + \frac{1}{\lambda_n^2 e^{\lambda_n^2 t}}\right).$$

Our total solution is then

$$20 + \sum_{n\geq 1} c_n(t) X_n(x).$$

1.1.2. *Variations 2.* Solve:

$$u_{tt} - u_{xx} = tx, \quad 0 < x < 4, \quad t \geq 0,$$

$$\begin{aligned} u(0,t) &= 20, \\ u_x(4,t) &= 0, \\ u(x,0) &= 20, \\ u_t(x,0) &= 0. \end{aligned}$$

Almost déjà vu right? Mais pas precisement... We have here an inhomogeneous wave equation. However, the inhomogeneity is *time dependent.* So, a steady state solution ain't gonna solve that problem. Next, we look at our boundary and initial conditions. The constant function, 20, satisfies that vertical list of conditions. So, we look for a function v to satisfy the inhomogeneous wave equation *but* with homogeneous BC and IC, thus we want v to satisfy

$$v_{tt} - v_{xx} = tx,$$

and $v(0,t) = v_x(4,t) = v(x,0) = v_t(x,0) = 0$. Our solution will be $u = 20 + v$. To solve the inhomogeneous heat equation, we will use the Fourier series method (Fourier series because on a bounded interval). The inhomogeneous part of the heat equation can be expressed using an L^2 OB $\{\phi_n\}$ for $[0,4]$ which satisfies the boundary condition and the SLP,

$$\phi_n''(x) + \lambda_n \phi_n(x) = 0, \quad \phi_n(0) = \phi_n'(4) = 0.$$

I leave it to you to check that the only λ_n for which there is such a $\phi_n \not\equiv 0$ are positive λ_n. The corresponding ϕ_n is thus a linear combination of sine and cosine, and to satisfy the BC at $x = 0$, we see that the cosine is out. So, we need a sine. In order to get the BC at $x = 4$, we need (up to multiplication by a factor which is constant with respect to x)

$$\phi_n = \sin((2n+1)\pi x/8), \quad \lambda_n = \frac{(2n+1)^2\pi^2}{64}.$$

Next, we shall allow the constant factor multiplying ϕ_n, to depend on time, and we write

$$v(t,x) = \sum_{n \in \mathbb{N}} c_n(t)\phi_n(x).$$

We can also express the xt side of the wave equation using the L^2 OB,

$$tx = t\sum_{n\geq 0} \widehat{x_n}\phi_n(x),$$

where

$$\widehat{x_n} = \frac{1}{2}\int_0^4 x\sin((2n+1/)\pi x/8)dx = \frac{8(-1)^n}{(n+1/2)^2}.$$

Next, we apply the wave operator to the expression for v in order to determine the unknown coefficient functions, c_n,

$$v_{tt} + v_{xx} = \sum_{n\geq 0} c_n''(t)\phi_n(x) - c_n(t)\phi_n''(x) = \sum_{n\geq 0}\left(c_n''(t) + \lambda_n c_n(t)\right)\phi_n(x).$$

We want this to equal

$$tx = \sum_{n\geq 0} t\widehat{x_n}\phi_n(x).$$

To obtain the equality, we equate the individual terms in each series, writing

$$(c_n''(t) + c_n(t)\lambda_n)\phi_n(x) = t\widehat{x_n}\phi_n(x).$$

Hence, we want c_n to satisfy the ODE:

$$c_n''(t) + \lambda_n c_n(t) = t\widehat{x_n}.$$

The homogeneous ODE

$$f'' + \lambda f = 0, \quad \lambda > 0, \quad \implies f(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$$

A particular solution to the inhomogeneous ODE is a function of the form

$$c(t) = at + b.$$

Substituting such a function into the ODE, we see that we need

$$c_n(t) = \frac{t\widehat{x_n}}{\lambda_n}.$$

Now we gotta look at the ICs. You see, the ϕ_n's take care of the BC's because we built them that way. However, they don't depend on time, so they can't help us with the ICs. We need the $c_n(t)$ to do that. Now, if we just take the particular solution to the ODE, we see that it vanishes at $t = 0$. However, we also want the derivative to vanish at $t = 0$, and it don't do that. So, we combine the particular solution with a solution to the homogeneous ODE. Hence, we want

$$c_n(t) = \frac{t\widehat{x_n}}{\lambda_n} + a_n\cos(\sqrt{\lambda_n}x) + b_n\sin(\sqrt{\lambda_n}x).$$

To make sure $c_n(0) = 0$ we need $a_n = 0$. To make sure $c_n'(0) = 0$, we need

$$\frac{\widehat{x_n}}{\lambda_n} + \sqrt{\lambda_n}b_n = 0 \implies b_n = -\frac{\widehat{x_n}}{\lambda_n^{3/2}}.$$

Hence, our full solution is

$$u(x,t) = 20 + \sum_{n\in\mathbb{N}}\left(\frac{t\widehat{x_n}}{\lambda_n} - \frac{\widehat{x_n}}{\lambda_n^{3/2}}\sin(\sqrt{\lambda_n}x)\right)\phi_n(x),$$

with

$$\lambda_n = \frac{(2n+1)^2\pi^2}{64}, \quad \widehat{x_n} = \frac{8(-1)^n}{(n+1/2)^2}, \quad \phi_n(x) = \sin((2n+1)\pi x/8).$$