FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Let us briefly enter two space dimensions. We will do a rather interesting example, solving

$$\Box u = 0 \text{ inside a rectangle, } u(x,y,0) = f(x,y), \quad u_t(x,y,0) = 0,$$

$$u(x,y,t) = g(x,y) \text{ for } (x,y) \text{ on the boundary of the rectangle.}$$

Here we have horrible inhomogeneous boundary condition. So, to solve the problem, we break it into two smaller problems which we tackle *one at a time.*

First, we see that the horrible inhomogeneous boundary condition is *time independent.* Hence, we can solve it by finding a steady state solution. So, we are looking for a function

$$\Phi(x,y)$$

to satisfy

$$\Box \Phi = 0 \text{ inside the rectangle,}$$

$$\Phi = g \text{ on the boundary of the rectangle.}$$

Since the physical problem doesn't care where in space the rectangle is sitting, let us put it so that its vertices are at $(0,0)$, $(0,B)$, $(A,0)$, (A,B). The BCs are terrible, so we deal with them one at a time. First, let's make nice BCs left and right, so we set

$$\phi(0,y) = \phi(A,y) = 0,$$

and we deal with horrible BCs up and down:

$$\phi(x,0) = g(x,0), \quad \phi(x,B) = g(x,B).$$

Let's try separation of variables:

$$-X''Y - Y''X = 0 \implies -\frac{Y''}{Y} = \frac{X''}{X} = \lambda.$$

The BCs for X are $X(0) = X(A) = 0$. We have solved this problem. The solutions are

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right)\sqrt{\frac{2}{A}}, \quad \lambda_n = -\frac{n^2\pi^2}{A^2}.$$

This means that the partner function

$$Y_n(y) = a_n \cosh\left(\frac{n\pi y}{A}\right) + b_n \sinh\left(\frac{n\pi y}{A}\right).$$

We smash them all together, writing

$$\phi(x,y) = \sum_{n\geq 1} X_n(x) Y_n(y).$$

This is okay because the PDE is homogeneous. Now, to get the horrible BCs, we need

$$\phi(x,0) = g(x,0) = \sum_{n\geq 1} a_n X_n(x).$$

Hence, the coefficients

$$a_n = \langle g(x,0), X_n\rangle = \int_0^A g(x,0)\overline{X_n(x)}dx.$$

For the other BC, we need

$$\phi(x,B) = g(x,B) = \sum_{n\geq 1} X_n(x)\left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right)\right).$$

Therefore we need

$$\begin{aligned}\left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right)\right) &= \langle g(x,B), X_n\rangle \\ &= \int_0^A g(x,B) X_n(x) dx.\end{aligned}$$

Solving for

$$b_n = \frac{\langle g(x,B), X_n\rangle - a_n \cosh\left(\frac{n\pi B}{A}\right)}{\sinh\left(\frac{n\pi B}{A}\right)}.$$

We can proceed analogously to deal with the other horrible BCs left and right, with nice BCs up and down:

$$\psi(x,0) = \psi(x,B) = 0, \quad \psi(0,y) = g(0,y), \quad \psi(A,y) = g(A,y).$$

By symmetry, the solution will be given by

$$\sum_{n\geq 1} \widetilde{X_n}(y)\widetilde{Y_n(x)},$$

with

$$\widetilde{X_n(y)} = \sin\left(\frac{n\pi y}{B}\right)\sqrt{\frac{2}{B}},$$

and

$$\widetilde{Y_n(x)} = \widetilde{a_n} \cosh\left(\frac{n\pi x}{B}\right) + \widetilde{b_n} \sinh\left(\frac{n\pi x}{B}\right).$$

The coefficients come from the horrible boundary conditions,

$$\widetilde{a_n} = \langle g(0,y), \widetilde{X_n}\rangle = \int_0^B g(0,y)\widetilde{X_n(y)}dy.$$

The other one

$$\widetilde{b_n} = \frac{\langle g(A,y), \widetilde{X_n}\rangle - \widetilde{a_n} \cosh\left(\frac{n\pi A}{B}\right)}{\sinh\left(\frac{n\pi A}{B}\right)}.$$

So, we have found

$$\psi(x,y) = \sum_{n\geq 1} \widetilde{X_n(y)}\widetilde{Y_n(x)}.$$

The full solution to this part of the problem is

$$\Phi(x,y) = \phi(x,y) + \psi(x,y).$$

Phew, we have dealt with the horrible (yet time independent) boundary condition by finding our steady state solution. Now, we just need to solve the lovely IVP for the wave equation with the Dirichlet boundary condition,

$$\Box u = 0, \quad u_t(x,y,0) = 0, \quad u(x,y,0) = f(x,y) - \Phi(x,y), \quad u = 0 \text{ on the boundary.}$$

We use separation of variables for t, x, and y. Write

$$u = TXY.$$

The PDE is

$$T'XY - X''TY - Y''TX = 0 \iff \frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda.$$

Consider the stuff with X and Y,

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda \implies \frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu.$$

You see, the same reasoning again says both sides are constant. So, we turn to our old friendly equation

$$X'' = \mu X, \quad X(0) = X(A) = 0.$$

We have solved this before. The solutions are

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right)\sqrt{\frac{2}{A}}, \quad \mu_n = -\frac{n^2\pi^2}{A^2}.$$

This gives the equation for Y,

$$\frac{Y''}{Y} = \lambda - \mu_n, \quad Y(0) = Y(B) = 0.$$

Let us briefly call

$$\nu = \lambda - \mu_n.$$

Then, the ODE

$$Y'' = \nu Y, \quad Y(0) = Y(B) = 0$$

has solutions

$$Y_m(y) = \sin\left(\frac{m\pi y}{B}\right)\sqrt{\frac{2}{B}} \quad \nu_m = -\frac{m^2\pi^2}{B^2}.$$

Since

$$\nu_m = \lambda - \mu_n \implies \lambda = \lambda_{n,m} = \nu_m + \mu_n = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}.$$

Recalling the equation for the partner function, T, we have

$$T_{n,m}(t) = a_{n,m}\cos(\sqrt{|\lambda_{n,m}|}t) + b_{n,m}\sin(\sqrt{|\lambda_{n,m}|}t).$$

Hence we write

$$u(x,y,t) = \sum_{n,m\geq 1} T_{n,m}(t)X_n(x)Y_m(y).$$

The initial condition

$$u_t(x,y,0) = 0 \implies b_{n,m} = 0\forall n,m.$$

The other condition is that

$$u(x,y,0) = f(x,y) - \Phi(x,y) = \sum_{n,m \geq 1} a_{n,m} X_n(x) Y_m(y).$$

Hence we require

$$a_{n,m} = \langle f - \Phi, X_n Y_m \rangle = \int_{[0,A]\times[0,B]} (f(x,y) - \Phi(x,y)) X_n(x) Y_m(y) dx dy.$$

The full solution is then

$$u(x,y,t) - \Phi(x,y).$$

Remark 1. The eigenvalues of the two-dimensional SLP we solved above,

$$\lambda_{n,m} = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}$$

are interesting to compare to the analogous one-dimensional case. In the analogous one dimension case, where we have

$$\mu_n = -\frac{n^2\pi^2}{A^2},$$

you can see that these are all square integer multiples of

$$\mu_1 = -\frac{\pi^2}{A^2}.$$

This is the mathematical reason that vibrating strings sound lovely. On the other hand, as long as the rectangle is not a square, that is $A \neq B$, it is no longer true that the $\lambda_{n,m}$ are all multiples of

$$\lambda_{1,1} = -\frac{\pi^2}{B^2} - \frac{\pi^2}{A^2}.$$

For this reason, vibrating rectangles can sound rather awful. You can listen to something along these lines (okay it's for tori not rectangles, but mathematically basically the same) here: `http://www.toroidalsnark.net/som.html`. Further exploration of the mathematics of music could make for an interesting bachelor's or master's thesis....

1.1. **The Fourier Transform.** With that musical interlude, it is now time to COMPLETELY CHANGE GEARS. We are now going to deal with FUNCTIONS AND PROBLEMS ON THE WHOLE REAL LINE. Why am I shouting? Well, it's because the techniques on finite intervals and those on $\mathbb{R}$ (or $\mathbb{R}^2$, $\mathbb{R}^n$ etc) are DIFFERENT. DO NOT MIX THEM UP. It's like that South Park episode with the pig and the elephant, "Pig and elephant just don't splice" `https://www.youtube.com/watch?v=RztfjHdM-pg`. The pig could be compared to a finite interval, whereas the elephant is $\mathbb{R}$. They just don't splice. They need to be considered separately.

We begin with

Definition 1. The set

$$\mathcal{L}^1(\mathbb{R}) = \text{ the set of equivalence classes, } [f] \text{ of functions which satisfy:}$$

$$f \text{ is measurable, and } \int_{\mathbb{R}} |f(x)| dx < \infty.$$

The function g belongs to the same equivalence class as f if $g = f$ almost everywhere on $\mathbb{R}$. Above, dx means the Lebesgue integral. If f is also Riemann integrable, then these two integrals are equal.

Next we have

Definition 2. The set

$$\mathcal{L}^2(\mathbb{R}) = \text{ the set of equivalence classes, } [f] \text{ of functions which satisfy:}$$

$$f \text{ is measurable, and } \int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

The function g belongs to the same equivalence class as f if $g = f$ almost everywhere on $\mathbb{R}$.

The space $H = \mathcal{L}^2(\mathbb{R})$ is a Hilbert space with the scalar product:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} d\mu.$$

Hence, by definition, the norm on $\mathcal{L}^2(\mathbb{R})$ is

$$||f||_{\mathcal{L}^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 dx}.$$

A lot of things which are true for $\mathcal{L}^2$ on a finite interval are no longer true on $\mathcal{L}^2(\mathbb{R})$. For example, the functions

$$e^{inx}, \sin(x), \cos(x)$$

are all neither in $\mathcal{L}^1(\mathbb{R})$ nor in $\mathcal{L}^2(\mathbb{R})$. Furthermore, there is no relationship between $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$. There are functions which are in $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sqrt{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

is in $\mathcal{L}^1(\mathbb{R})$ but it is *not* in $\mathcal{L}^2(\mathbb{R})$. Go ahead and compute the integrals! On the other hand, the function

$$f(x) = \begin{cases} 0 & x \leq 1 \\ \frac{1}{x} & x > 1 \end{cases}$$

is in $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$. The function

$$e^{-x}$$

is in both $\mathcal{L}^1(\mathbb{R})$ and in $\mathcal{L}^2(\mathbb{R})$. So, all we can say is that

$$\mathcal{L}^1(\mathbb{R}) \not\subset \mathcal{L}^2(\mathbb{R}), \quad \mathcal{L}^2(\mathbb{R}) \not\subset \mathcal{L}^1(\mathbb{R}), \quad \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R}) \neq \emptyset.$$

So, we're in a whole new territory here. We will first define the Fourier transform of a function, denoted by

$$\hat{f}(\xi)$$

for functions in $\mathcal{L}^1(\mathbb{R})$. We'll see how to extend to $\mathcal{L}^2(\mathbb{R})$ later....

Proposition 3. *Assume that* $f \in \mathcal{L}^1(\mathbb{R})$. *Then*

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

is a well-defined complex number for any $\xi \in \mathbb{R}$.

Proof: Simply estimate

$$\left| \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \right| \leq \int_{\mathbb{R}} |f(x)| dx < \infty.$$

We're going to be using something called the *convolution.*

Definition 4. The convolution of f and g is a function $f * g : \mathbb{R} \to \mathbb{C}$ defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-y) g(y) dy,$$

whenever the integral on the right exist.

Proposition 5. *Assume that f and g are both in $\mathcal{L}^2(\mathbb{R})$. Then*
(1) $|f * g(x)| \leq ||f||||g||$ for all $x \in \mathbb{R}$
(2) $f * (ag + bh) = af * g + bf * h$ for all $a, b \in \mathbb{C}$
(3) $f * g = g * f$
(4) $f * (g * h) = (f * g) * h$

Proof: This is useful to do because it helps to familiarize oneself with the convolution. We first estimate

$$|f * g(x)| = \left| \int_{\mathbb{R}} f(x-y) g(y) dy \right| \leq \int_{\mathbb{R}} |f(x-y)||g(y)| dy.$$

The point $x \in \mathbb{R}$ is fixed and arbitrary, so I define a function

$$\phi(y) = f(x-y).$$

Then

$$|f * g(x)| \leq \int_{\mathbb{R}} |\phi(y)||g(y)| dy \leq ||\phi||||g||.$$

We compute

$$||\phi||^2 = \int_{\mathbb{R}} |f(x-y)|^2 dy = -\int_{\infty}^{-\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(t)|^2 dt = ||f||^2.$$

Above, we used the substitution $t = x - y$ so $dt = -dy$, and the integral got reversed. The $-$ goes away when we re-reverse the integral. So, in the end we see that

$$|f * g(x)| \leq ||f||||g||$$

as desired. The second property follows simply by the linearity of the integral itself. For the third property, we will use substitution again:

$$f * g(x) = \int_{\mathbb{R}} f(x-y) g(y) dy.$$

We want to get $g(x-z)$ so we define

$$y = x - z \implies x - y = z, \quad dz = -dy.$$

Hence,

$$f * g(x) = -\int_{\infty}^{-\infty} f(z) g(x-z) dz = \int_{-\infty}^{\infty} g(x-z) f(z) dz = g * f(x).$$

We do something rather similar in the fourth property:

$$f * (g * h)(x) = \int_{\mathbb{R}} f(x-y) \int_{\mathbb{R}} g(y-z)h(z)dzdy.$$

For the other term we have

$$(f * g) * h(x) = \int_{\mathbb{R}} (f * g)(x-y)h(y)dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y-z)g(z)h(y)dzdy.$$

So, we define

$$t = y - z \implies x - y = x - t - z, \quad dt = dy.$$

Then

$$f * (g * h)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t-z)g(t)h(z)dzdt.$$

Finally, we call $z = y$ and $t = z$ (sorry if this gives you a headache!) because they are just names, and then we get

$$f * (g * h)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y-z)g(y)h(z)dzdy.$$

If you're worried about the order of integration, don't be. Since everything is in $\mathcal{L}^2$, these integrals converge absolutely, so those Italian magicians, Fubini & Tonelli allow us to do the switch-a-roo with the integrals as much as we like.

There is a giant theorem about approximations using the convolution, but let's not get ahead of ourselves just yet. We shall save that for later after we've done some more basic things with the Fourier transform.

Proposition 6 (Mollification)**.** *If $f \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$, $f' \in \mathcal{L}^2(\mathbb{R})$, and $g \in \mathcal{L}^2(\mathbb{R})$, then $f * g \in \mathcal{C}^1(\mathbb{R})$. Moreover $(f * g)' = f' * g$.*

Proof: Everything converges beautifully so just stick that differentiation right under the integral defining

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Hence

$$(f * g)'(x) = \int_{\mathbb{R}} f'(x-y)g(y)dy = f' * g(x).$$

If you are not satisfied with this explanation, a rigorous proof can be obtained using the Dominated Convergence Theorem, but that is a theorem which we cannot prove in the context of this humble course.

So, the idea is that if you convolve some rough g with a nice, smooth, f, then f *mollifies* g. This means that g inherits the smoothness from f. We will see later that the initial value problem for the heat equation on $\mathbb{R}$,

$$u_t - u_{xx} = 0, \quad u(x,0) = f(x) \in \mathcal{L}^2(\mathbb{R})$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy.$$

So, you see that for $t > 0$, even though the initial data was just in $\mathcal{L}^2(\mathbb{R})$ (so it could be nowhere differentiable, for example), the solution $u(x,t)$ to the heat equation is smooth for all $t > 0$. That's because the Gaussian term, $e^{-y^2/(4t)}$ is fantastic. It and all its derivatives are in $\mathcal{L}^2(\mathbb{R})$. It is of course smooth. So, we can do the proposition a zillion times and get as many derivatives of u as we like. This is what I mean by the solution operator to the heat operator being a "smoothing operator." Similarly, when I told you about infinite speed of propagation, assume that the initial data satisfies $f \geq 0$, and there exists a constant ε and a set with some positive length such that $f \geq \varepsilon > 0$ on this set. Then, for all $t > 0$

$$u(x,t) > 0 \forall x \in \mathbb{R}.$$

So, for example if my initial data satisfies

$$f(x) = \begin{cases} \varepsilon & x \in (-0.0000001, 0.0000001) \\ 0 & \text{otherwise} \end{cases}$$

for some tiny $\varepsilon > 0$ like maybe $\varepsilon = 10^{-11111}$, then the solution to the heat equation, $u(x,t)$ with this initial data is positive everywhere for $t > 0$. So, it's like that little teeny tiny bit of heat shoots out instantaneously over the whole real line! That's pretty neat. It's also why if you are ever in a fire situation, you should just GET OUT. Don't look for your stuff or whatever, just get yourself (and if you got a baby or somebody else who needs help or a pet - grab them too - just don't worry about the non-living stuff) OUT. Cause heat has infinite speed of propagation. Also, at the particle level it corresponds to random motion. So, random and infinite speed of propagation means just beat it!

Let us continue with the Fourier transform, which we will use to prove the solution to the IVP for the homogeneous heat equation is indeed as above.

Theorem 7 (Properties of the Fourier transform)**.** *Assume that everything below is well defined. Then, the Fourier transform,*

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

satisfies

(1) $\mathcal{F}(f(x-a))(\xi) = e^{-ia}\hat{f}(\xi)$.
(2) $\mathcal{F}(f')(\xi) = i\xi\hat{f}(\xi)$
(3) $\mathcal{F}(xf(x))(\xi) = i\mathcal{F}(f)'(\xi)$
(4) $\mathcal{F}(f * g)(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

Proof: We just compute (we are being a bit naughty, not bothering with issues of convergence, but all such issues are indeed rigorously verifiable, so not to worry). First

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(x-a) e^{-ix\xi} dx.$$

Change variables. Let $t = x - a$, then $dt = dx$, and $x = t + a$ so

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(t) e^{-i(t+a)\xi} dt = e^{-ia\xi}\hat{f}(\xi).$$

The next one will come from integrating by parts:

$$\int_{\mathbb{R}} f'(x) e^{-ix\xi} dx = f(x) e^{-ix\xi} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} -i\xi f(x) e^{-ix\xi} dx = i\xi\hat{f}(\xi).$$

The boundary terms vanish because of reasons (again it is $\mathcal{L}^1$ and $\mathcal{L}^2$ theory stuff). Similarly we compute

$$\int_{\mathbb{R}} xf(x)e^{-ix\xi}dx = -\frac{1}{i}\int_{\mathbb{R}} f(x)\frac{d}{d\xi}e^{-ix\xi}dx = i\frac{d}{d\xi}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx = i\mathcal{F}(f)'(\xi).$$

Finally,

$$\mathcal{F}(f*g)(\xi) = \int_{\mathbb{R}} f*g(x)e^{-ix\xi}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-ix\xi}dydx.$$

We do a little sneaky trick

$$\begin{aligned} &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-ix\xi}e^{-iy\xi}e^{iy\xi}dydx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)e^{-i(x-y)\xi}g(y)e^{-iy\xi}dydx. \end{aligned}$$

Let $z = x - y$. Then $dz = -dy$ so

$$\begin{aligned} = \int_{\mathbb{R}}\int_{\infty}^{-\infty} f(z)e^{-iz\xi}(-dz)g(y)e^{-iy\xi}dy &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(z)e^{-iz\xi}dzg(y)e^{-iy\xi}dy \\ &= \hat{f}(\xi)\hat{g}(\xi). \end{aligned}$$