FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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We begin with a theorem.

Theorem 1 (Plancharel)**.** *There is a well defined unique extension of the Fourier transform to $\mathcal{L}^2(\mathbb{R})$. For any $f \in \mathcal{L}^2(\mathbb{R})$, $\hat{f} \in \mathcal{L}^2(\mathbb{R})$. Moreover,*

$$\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle,$$

and thus

$$||\hat{f}||^2_{\mathcal{L}^2} = 2\pi ||f||^2,$$

for all f and g in $\mathcal{L}^2(\mathbb{R})$.

Proof: Start with the left side and use the FIT on f, to write

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \overline{g(x)} d\xi dx.$$

Move the complex conjugate to engulf the $e^{ix\xi}$,

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \overline{g(x) e^{-ix\xi}} d\xi dx.$$

Swap the order of integration and integrate x first:

$$\int_{\mathbb{R}} \hat{f}(\xi) \overline{g(x) e^{-ix\xi}} dx = \hat{f}(\xi) \overline{\hat{g}(\xi)}.$$

Put it back in:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle.$$

We may from time to time use the following cute fact as well.

Lemma 2 (Riemann & Lebesgue)**.** *Assume $f \in \mathcal{L}^1(\mathbb{R})$. Then,*

$$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0.$$

We shall indeed need to actually prove the next one, because it's going to be quite important for the initial value problem for the heat equation.

1.1. **The big bad convolution approximation theorem.** This theory item is Theorem 7.3, regarding approximation of a function by convoluting it with a so-called "approximate identity." This theorem and its proof are both rather long. The proof relies very heavily on knowing the definition of limits and how to work with those definitions, so if you're not comfortable with ϵ and δ style arguments, it would be advisable to brush up on these.

Theorem 3. *Let $g \in L^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} g(x)dx = 1.$$

Define

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\varepsilon > 0$,

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof. We would like to show that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy = \alpha f(x+) + \beta f(x-)$$

which is equivalent to showing that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \alpha f(x+) - \beta f(x-) = 0.$$

We now insert the definitions of α and β, so we want to show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

We can prove this if we show that

$$\lim_{\varepsilon \to 0} \int_{-\infty} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy = 0$$

and also

$$\lim_{\varepsilon \to 0} \int_{0}^{\infty} f(x-y) g_\varepsilon(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

In the textbook, Folland proves that the second of these holds. So, for the sake of diversity, we prove that the first of these holds. The argument is the same for both, so proving one of them is sufficient.

Hence, we would like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^{0} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy$$

as small as we like. To make this precise, let us assume that "as small as we like" is quantified by a very small $\delta > 0$. This is part of the whole points of limits. This quantity,

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy$$

can be made arbitrarily small by choosing ε sufficiently small, but it may never actually vanish as long as $\epsilon > 0$. You'll just need to sit for a while and think about that, to make sure you're really comfortable with the concept of limits...

Proceeding with the proof, we smash the two integrals together, writing

$$\int_{-\infty}^{0} \left(f(x-y)g_\varepsilon(y) - f(x+)g(y)\right) dy.$$

Well, this is a bit inconvenient, because in the first part we have g_ε, but in the second part it's just g. So, we make a small observation,

$$\int_{-\infty}^{0} g(y)dy = \int_{-\infty}^{0} g(z/\varepsilon)\frac{dz}{\varepsilon} = \int_{-\infty}^{0} g_\varepsilon(z)dz$$

Above, we have made the substitution $z = \varepsilon y$, so $y = z/\varepsilon$, and $dz/\varepsilon = dy$. The limits of integration don't change. By this calculation,

$$\int_{-\infty}^{0} f(x+)g(y)dy = \int_{-\infty}^{0} f(x+)g_\varepsilon(y)dy.$$

(Above the integration variable was called z, but what's in a name? The name of the integration variable doesn't matter!). Moreover, note that $f(x+)$ is a constant, so it's just sitting there doing nothing. Hence, we have computed that

$$\int_{-\infty}^{0} \left(f(x-y)g_\varepsilon(y) - f(x+)g(y)\right) dy = \int_{-\infty}^{0} g_\varepsilon(y)\left(f(x-y) - f(x+)\right) dy.$$

Remember that $y \le 0$ where we're integration. Therefore, $x - y \ge x$. Moreover, by definition

$$\lim_{y \uparrow 0} f(x-y) = f(x+) \implies \lim_{y \uparrow 0} f(x-y) - f(x+) = 0.$$

By definition of limit (if you're not comfortable with this definition by now, you really need to get comfortable with it!) there exists $y_0 < 0$ such that for all $y \in (y_0, 0)$

$$|f(x-y) - f(x+)| < \widetilde{\delta}.$$

We are using $\widetilde{\delta}$ for now, to indicate that $\widetilde{\delta}$ is going to be something in terms of δ, engineered in such a way that at the end of our argument we get that for ϵ sufficiently small,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y)\left(f(x-y) - f(x+)\right) dy \right| < \delta.$$

So, to figure out this $\widetilde{\delta}$, we use our estimate on the part of the integral from y_0 to 0,

$$\left| \int_{y_0}^{0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \le \int_{y_0}^{0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy$$

$$\le \widetilde{\delta} \int_{y_0}^{0} |g_\varepsilon(y)|dy \le \widetilde{\delta} \int_{\mathbb{R}} |g_\varepsilon(y)|dy = \widetilde{\delta}||g||.$$

Above, we have used the same substitution trick to see that

$$\int_{\mathbb{R}} |g_\varepsilon(y)|dy = \int_{\mathbb{R}} |g(z)|dz = ||g||,$$

where $||g||$ is the $L^1(\mathbb{R})$ norm of g. By assumption, $g \in L^1(\mathbb{R})$, so this L^1 norm is finite. Moreover, because we know that

$$\int_{\mathbb{R}} g(y)dy = 1,$$

we know that

$$||g|| = \int_{\mathbb{R}} |g(y)|dy \geq \left| \int_{\mathbb{R}} g(y)dy \right| = 1.$$

Hence, I propose setting

$$\widetilde{\delta} = \frac{\delta}{2||g||}.$$

Note that we're not dividing by zero, by the above observation that $||g|| \geq 1$. So, this is a perfectly decent number. Then, we have the estimate (repeating the above estimate)

$$\left| \int_{y_0}^{0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \leq \int_{y_0}^{0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy$$

$$\leq \widetilde{\delta} \int_{y_0}^{0} |g_\varepsilon(y)|dy \leq \widetilde{\delta} \int_{\mathbb{R}} |g_\varepsilon(y)|dy = \widetilde{\delta}||g|| = \frac{\delta}{2}.$$

To complete the proof, we just need to estimate the other part of the integral, from $-\infty$ to y_0. It is important to remember that

$$y_0 < 0.$$

So, we wish to estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right|.$$

Here we need to consider the two possible cases given in the statement of the theorem separately. First, let us assume that f is bounded, which means that there exists $M > 0$ such that $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$. Hence

$$|f(x-y) - f(x+)| \leq |f(x-y)| + |f(x+)| \leq 2M.$$

So, we have the estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \leq \int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy \leq 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)|dy.$$

We shall do a substitution now, letting $z = y/\varepsilon$. Then, as we have computed before,

$$\int_{-\infty}^{y_0} |g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon} |g(z)|dz.$$

Here the limits of integration do change, because $y_0 < 0$. Specifically $y_0 \neq 0$, which is why the top limit changes. Now, let's think about what happens as $\varepsilon \to 0$. We're integrating between $-\infty$ and y_0/ε. We know that $y_0 < 0$. So, when we divide it by a really small, but still positive number, like ε, then $y_0/\varepsilon \to -\infty$ as $\varepsilon \to 0$. Moreover, we know that

$$\int_{-\infty}^{0} |g(y)| dy < \infty.$$

What this really means is that

$$\lim_{R \to -\infty} \int_{R}^{0} |g(y)| dy = \int_{-\infty}^{0} |g(y)| dy < \infty.$$

Hence,

$$\lim_{R \to -\infty} \int_{-\infty}^{0} |g(y)| dy - \int_{R}^{0} |g(y)| dy = 0.$$

Of course, we know what happens when we subtract the integral, which shows that

$$\lim_{R \to -\infty} \int_{-\infty}^{R} |g(y)| dy = 0.$$

Since

$$\lim_{\varepsilon \to 0} y_0/\varepsilon = -\infty,$$

this shows that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{y_0/\varepsilon} |g(y)| dy = 0.$$

Hence, by definition of limit (see, here it comes again), there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{-\infty}^{y_0/\varepsilon} |g(y)| dy < \frac{\delta}{4(M+1)}.$$

Then, combining this with our estimates, above, which we repeat here,

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+)) g_\varepsilon(y) dy \right| \le \int_{-\infty}^{y_0} |f(x-y) - f(x+)| |g_\varepsilon(y)| dy \le 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)| dy$$

$$< 2M \frac{\delta}{4(M+1)} < \frac{\delta}{2}.$$

Therefore, we have the estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right|$$

$$\le \int_{-\infty}^{0} |g_\varepsilon(y)| |f(x-y) - f(x+)| dy \le \int_{-\infty}^{y_0} |f(x-y) - f(x+)| |g_\varepsilon(y)| dy + \int_{y_0}^{0} |f(x-y) - f(x+)| |g_\varepsilon(y)| dy$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Finally, we consider the other case in the theorem, which is that g vanishes outside a bounded interval. We retain the first part of our estimate, that is

$$\int_{y_0}^{0} |f(x-y) - f(x+)| |g_\varepsilon(y)| dy < \frac{\delta}{2}.$$

Next, we again observe that

$$\lim_{\varepsilon \downarrow 0} \frac{y_0}{\varepsilon} = -\infty.$$

By assumption, we know that there exists some $R > 0$ such that

$$g(x) = 0 \forall x \in \mathbb{R} \text{ with } |x| > R.$$

Hence, we may choose ε sufficient small so that

$$\frac{y_0}{\varepsilon} < -R.$$

Specifically, let

$$\varepsilon_0 = \frac{1}{-Ry_0} > 0.$$

Then for all $\varepsilon \in (0, \varepsilon_0)$ we compute that

$$\frac{y_0}{\varepsilon} < -R.$$

Hence for all $y \in (-\infty, y_0/\varepsilon)$ we have $g(y) = 0$. Thus, we compute as before using the substitution $z = y/\varepsilon$,

$$\int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon} |f(x-\varepsilon z) - f(x+)||g(z)|dz = 0,$$

because $g(z) = 0 \forall z \in (-\infty, y_0/\varepsilon)$. Thus, we have the total estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right|$$

$$\leq \int_{-\infty}^{0} |g_\varepsilon(y)||f(x-y)-f(x+)|dy \leq \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$< 0 + \frac{\delta}{2} \leq \delta.$$

□

As a consequence we can obtain the Fourier inversion theorem.

Theorem 4 (Fourier inversion theorem)**.** *Assume that $f \in \mathcal{L}^1(\mathbb{R})$ and is piecewise continuous on $\mathbb{R}$. Assume that at its points of discontinuity*

$$f(x) = \frac{1}{2} \left(f(x_-) + f(x_+) \right).$$

Then

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi.$$

Moreover, if $\hat{f} \in \mathcal{L}^1(\mathbb{R})$, then f is continuous and

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi.$$

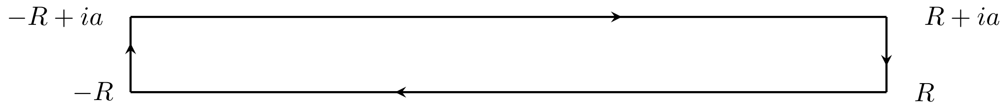
Finally, if $f \in \mathcal{L}^2(\mathbb{R})$, then the equality

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi$$

holds for almost every $x \in \mathbb{R}$.

Proof: The first part of the theorem is an application of the big bad convolution theorem. Let's just write out the integral on the right

$$\int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi.$$



box

Figure 1. The contour over which we integral. Call the contour Γ.

Using the definition of the Fourier transform, this is

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi}e^{-iy\xi}e^{-\varepsilon^2\xi^2/2}f(y)dyd\xi = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\xi(x-y)}e^{-\varepsilon^2\xi^2/2}f(y)dyd\xi.$$

If we (to the chagrin of the theoretical mathematicians, but again, trust me, it's rigorously justifiable!) change the order of integration and do the ξ integral first, we are computing the Fourier transform of $e^{-\varepsilon^2\xi^2/2}$ evaluated at the point $y-x$. I will do this calculation once for fun. The idea is to complete the square in the exponent:

$$-\varepsilon^2\xi^2/2 + i\xi(x-y) = -\left(\frac{\varepsilon\xi}{\sqrt{2}} - \frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2 + \left(\frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2.$$

Therefore

$$\int e^{-\varepsilon^2\xi^2/2+i\xi(x-y)}d\xi = \int e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}}-\frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2+\left(\frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2}d\xi$$
$$= e^{-\frac{(x-y)^2}{2\varepsilon^2}}\int e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}}-\frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2}d\xi.$$

Using a teensy bit of complex analysis, we can throw away the imaginary part in the exponent (to see this, draw a box, look at the two sides, show they go to zero, function is holomorphic inside the box, hence integral on the Loooooong top and bottom sides of the box are same, sides of box tend to zero, voilá). Oh wait, I can actually draw a picture of this (finally, in some joint work I recently learned to use tikz). So, I claim that

Rbox

$$\lim_{R\to\infty}\int_{-R}^{R} e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}}-\frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2}d\xi = \lim_{R\to\infty}\int_{-R}^{R} e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}}\right)^2}d\xi. \tag{1.1}$$

We integrate over the box in Figure box 1. For the sake of simplicity, assume $y > x$. Let

$$a = \frac{(y-x)}{\varepsilon^2}.$$

We integrate the function

$$f(z) = e^{-\frac{\varepsilon^2}{2}z^2}.$$

At the bottom of the box, $z = \xi \in \mathbb{R}$, and

$$f(z) = f(\xi) = e^{-\frac{\varepsilon^2\xi^2}{2}}.$$

At the top of the box $z = \xi + ia$, where

$$a = \frac{(y-x)}{\varepsilon^2} \implies f(z) = f(\xi+ia) = \exp\left(-\frac{\varepsilon^2}{2}\left(\xi + i\frac{(y-x)}{\varepsilon^2}\right)^2\right)$$
$$= \exp\left(-\left(\frac{\varepsilon\xi}{\sqrt{2}} - i\frac{(x-y)}{\sqrt{2}\varepsilon}\right)^2\right).$$

Since the function $f(z)$ is super nice and holomorphic,

$$\int_\Gamma f(z)dz = 0.$$

Moreover, the left and right integrals have $|\Re(z)| = R$. So, there,

$$f(\pm R + i\Im z) = \exp\left(-\frac{\varepsilon^2}{2}(\pm R + i\Im z)^2\right) \leq \exp\left(-\frac{\varepsilon^2 R^2}{2}\right),$$

since $\Im z \geq 0$. Hence, the integral on the two sides are bounded above by

$$|a| \exp\left(-\frac{\varepsilon^2 R^2}{2}\right) \to 0 \text{ as } R \to \infty.$$

So, as we let $R \to \infty$, the integral over the contour remains zero, the integrals over the sides of the box tend to zero, and we indeed see that (1.1) is true.

Thus, we just need to compute

$$\int e^{-\xi^2\varepsilon^2/2} d\xi.$$

We use a substitution, $t = \xi\varepsilon/\sqrt{2}$ so $dt = \varepsilon d\xi/\sqrt{2}$, and

$$\int e^{-\xi^2\varepsilon^2/2} d\xi = \frac{\sqrt{2}}{\varepsilon} \int e^{-t^2} dt = \frac{\sqrt{2\pi}}{\varepsilon}.$$

Thus

$$\mathcal{F}(e^{-\varepsilon^2\xi^2/2})(y-x) = \int e^{-\varepsilon^2\xi^2/2 + i\xi(x-y)} d\xi = e^{-\frac{(x-y)^2}{2\varepsilon^2}} \frac{\sqrt{2\pi}}{\varepsilon}.$$

So, in summary, we have computed:

$$\int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2\xi^2/2} \hat{f}(\xi) d\xi = \frac{\sqrt{2\pi}}{\varepsilon} \int e^{-\frac{(x-y)^2}{2\varepsilon^2}} f(y) dy = \sqrt{2\pi} \int \frac{e^{-\frac{(x-y)^2}{2\varepsilon^2}}}{\varepsilon} f(y) dy.$$

To be continued...