FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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♡♡♡♡♡ HAPPY VALENTINE'S DAY! ♡♡♡♡♡

Don't be depressed if you don't have a "Valentine" today. We've at least got math! Let's return to where we left off last time...

Theorem 1 (Fourier inversion theorem)**.** *Assume that $f \in \mathcal{L}^1(\mathbb{R})$ and is piecewise continuous on $\mathbb{R}$. Assume that at its points of discontinuity*

$$f(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right).$$

Then

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi.$$

Moreover, if $\hat{f} \in \mathcal{L}^1(\mathbb{R})$, then f is continuous and

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi.$$

Finally, if $f \in \mathcal{L}^2(\mathbb{R})$, then the equality

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi$$

holds for almost every $x \in \mathbb{R}$.

Proof: (Continued) We computed

$$\int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2\xi^2/2} \hat{f}(\xi) d\xi = \frac{\sqrt{2\pi}}{\varepsilon} \int e^{-\frac{(x-y)^2}{2\varepsilon^2}} f(y) dy = \sqrt{2\pi} \int \frac{e^{-\frac{(x-y)^2}{2\varepsilon^2}}}{\varepsilon} f(y) dy.$$

Ignoring the constant in front, we recognize

$$\int \frac{e^{-\frac{(x-y)^2}{2\varepsilon^2}}}{\varepsilon} f(y) dy = g_\varepsilon \star f(x), \quad g(x) = e^{-\frac{x^2}{2}} \implies g_\varepsilon(x) = \frac{1}{\varepsilon} e^{-\frac{x^2}{2\varepsilon^2}}.$$

Now, we see that, letting $t = x/\sqrt{2}$ so $\sqrt{2}dt = dx$

$$\int_{\mathbb{R}} g(x) dx = \int_{\mathbb{R}} e^{-x^2/2} dx = \int_{\mathbb{R}} e^{-t^2} \sqrt{2} dt = \sqrt{2\pi}.$$

Moreover, g is an even function, so

$$\int_{-\infty}^{0} g(x)dx = \frac{\sqrt{2\pi}}{2} = \int_{0}^{\infty} g(x)dx.$$

Hence, the big bad convolution theorem says that

$$\lim_{\varepsilon \to 0} g_\varepsilon \star f(x) = \frac{\sqrt{2\pi}}{2} \left(f(x_-) + f(x_+) \right).$$

Therefore when we bring back the constant factor of $\sqrt{2\pi}$, we get

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi = \lim_{\varepsilon \to 0} \sqrt{2\pi} g_\varepsilon \star f(x) = \frac{2\pi}{2} \left(f(x_-) + f(x_+) \right).$$

So, dividing by 2π, we get

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi = \frac{1}{2} \left(f(x_-) + f(x_+) \right) = f(x).$$

So that's how we get the first part, because at points of continuity, $f(x)$ is the average of its left and right limits, and at points of discontinuity, we assumed it was defined to equal this average. For the second statement, we look at

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi.$$

Well, since the integrand is in $\mathcal{L}^1$ we can use the dominated convergence theorem (if we know what the heck that is) to bring the limit into the integral. When we do that we get

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi.$$

We already saw that this limit converges to $f(x)$. However, now to get the continuity we use this integral formula for f and the fact that $\hat{f}$ is in $\mathcal{L}^1$ to use the DCT again (if we know what the heck that is) to say

$$\lim_{\delta \to 0} f(x+\delta) = \lim_{\delta \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(x+\delta)} \hat{f}(\xi) d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \lim_{\delta \to 0} e^{i\xi(x+\delta)} \hat{f}(\xi) d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi$$
$$= f(x).$$

Don't be cross with me for taking the limit into the integral. It is JUSTIFIED by the DCT (dominated convergence theorem). Just rest assured that if you take Integration Theory, you will see that indeed, the above switch-a-roo of limits is rigorously valid.

The last statement for $f \in \mathcal{L}^2(\mathbb{R})$ could be shown using an approximation argument. If it is too theoretical, just ignore this part. Smooth, compactly supported (this means they are zero outside of a compact set) functions are dense in $\mathcal{L}^2$. So, we take a sequence of them ϕ_n with $\phi_n \to f$ in $\mathcal{L}^2$ norm. Smooth, compactly supported functions are Schwarz class, so their Fourier transforms are Schwarz class (hence in $\mathcal{L}^1$. Schwarz class means the function and all its derivatives decay faster than x^{-n} as $|x| \to \infty$ for any $n \in \mathbb{N}$. This is called rapidly decaying). Hence, for each of the ϕ_n, its Fourier transform is in $\mathcal{L}^1$ so we got

$$\phi_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{\phi}_n(x) dx.$$

We have

$$||\phi_n - f|| \to 0, \quad n \to \infty$$

because we assumed $\phi_n \to f$ in $\mathcal{L}^2$ norm. By Plancharel's theorem, we also have

$$||\hat{\phi}_n - \hat{f}|| \to 0, \quad n \to \infty$$

and therefore also

$$\int_{\mathbb{R}} |e^{ix\xi}\hat{\phi}_n(\xi) - e^{ix\xi}\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\hat{\phi}_n(\xi) - \hat{f}(\xi)|^2 d\xi = ||\hat{\phi}_n - \hat{f}||^2 \to 0, \quad n \to \infty.$$

This shows that

$$||\frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{\phi}_n(\xi)d\xi - \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}(\xi)d\xi|| \to 0, \quad n \to \infty.$$

So, since ϕ_n is given by this integral formula we have

$$\phi_n \to \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}(\xi)d\xi, \quad n \to \infty$$

as elements of $\mathcal{L}^2$, and also

$$\phi_n \to f, \quad n \to \infty$$

as elements of $\mathcal{L}^2$. By the uniqueness of limits in $\mathcal{L}^2$,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}(\xi)d\xi, \qquad \text{as elements of } \mathcal{L}^2.$$

This means (by definition of $\mathcal{L}^2$ that $f(x)$ is equal to that integral on the right for almost every x. That's good enough.

1.1. **Applications.** Let's see how to solve the IVP for the heat equation. This is pretty fun.

1.1.1. *IVP for the homogeneous heat equation.* We wish to find u to satisfy

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = v(x) \in \mathcal{L}^2(\mathbb{R}) \end{cases}$$

We hit the PDE with the Fourier transform IN THE x VARIABLE:

$$\hat{u}_t(\xi,t) - \hat{u}_{xx}(\xi,t) = 0.$$

Now, we use the theorem which gave us the properties of the Fourier transform. It says that if we take the Fourier transform of a derivative, $\hat{f'}(\xi) = i\xi\hat{f}(\xi)$. Using this twice,

$$\hat{u}_{xx}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

Now, those of you who are picky about switching limits may not like this, but it is in fact rigorously valid:

$$\partial_t\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t) = 0.$$

Hence

$$\partial_t\hat{u}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

Well this is a first order homogeneous ODE for u in the t variable. We can solve it!!! We do that and get

$$\hat{u}(\xi,t) = e^{-\xi^2 t}c(\xi).$$

The constant can depend on ξ but not on t. To figure out what the constant should be, we use the IC:

$$\hat{u}(\xi,0) = \hat{v}(\xi) \implies c(\xi) = \hat{v}(\xi).$$

Thus, we have found

$$\hat{u}(\xi, t) = e^{-\xi^2 t}\hat{v}(\xi).$$

Now, we use another property of the Fourier transform which says

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

So, if we can find a function whose Fourier transform is $e^{-\xi^2 t}$, then we can express u as a convolution of that function and v. So, we are looking to find

$$g(x,t) \text{ such that } \hat{g}(x,t) = e^{-\xi^2 t}.$$

We can use the inversion formula:

$$g(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-\xi^2 t}d\xi.$$

Now, we can compute this integral much like we did before. We complete the square in the exponent:

$$-\xi^2 t + ix\xi = -\left(\xi\sqrt{t} + \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}.$$

Therefore we are computing

$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} + \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right)d\xi.$$

By the same contour integral trick, we can just toss out that imaginary stuff. So, we compute (using a change of variables to $y = \xi\sqrt{t}$ so $t^{-1/2}dy = d\xi$)

$$\int_{\mathbb{R}} e^{-\xi^2 t}d\xi = \frac{1}{\sqrt{t}}\int_{\mathbb{R}} e^{-y^2}dy = \frac{\sqrt{\pi}}{\sqrt{t}}.$$

Hence,

$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} + \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right)d\xi = \frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{x^2}{4t}}.$$

Recalling the factor of $1/(2\pi)$ we have

$$g(x,t) = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{x^2}{4t}} = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}.$$

Hence the solution is

$$u(x,t) = g * v(x) = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}}e^{-(x-y)^2/(4t)}v(y)dy.$$

1.1.2. *Computing tricky integrals.* There are two things to keep in mind:

$$\hat{f}(0) = \int_{\mathbb{R}} f(x)dx.$$

So, if you have the integral of a function, check and see what that function's Fourier transform is. If the Fourier transform is simple to compute, awesome, you can evaluate it at 0 and get the value of the integral. For example,

$$\int_{\mathbb{R}} \frac{1}{x^2+9}dx.$$

We see this is # 10 in Folland's TABLE 2. It is inevitably in BETA somewhere also... On the right side, we get the Fourier transform (with $a = 3$) is given by

$$\frac{\pi}{3} e^{-3|\xi|}.$$

So, this integral is the Fourier transform with $\xi = 0$, hence the value of the integral is

$$\frac{\pi}{3}.$$

That was pretty easy right? For something more complicated, you could have say

$$\int_{\mathbb{R}} f(x) g(x) dx,$$

with some icky functions f and g (see extra övning # 9). Now, you can use that the Fourier transform of a product is

$$(2\pi)^{-1} (\hat{f} * \hat{g})(\xi).$$

Hence, what you have above is

$$\int_{\mathbb{R}} f(x) g(x) dx = \int_{\mathbb{R}} e^{-i(0)x} f(x) g(x) dx = (2\pi)^{-1} (\hat{f} * \hat{g})(0).$$

So, if the Fourier transforms of these functions are somewhat better than the functions f and g, then the stuff on the right could be nicely computable and give you the integral on the left. Try # 9 to see how this works. (If you get stuck, Team Fourier is here to help! Just ask us!)

As another example, there is extra exercise number 10. It says you know the Fourier transform of $f(t)$ is $\frac{1}{|w|^3+1}$. We're supposed to compute

$$\int_{\mathbb{R}} |f \star f'|^2 dt.$$

Yikes! Looks scary eh? Well, let's stay calm and carry on. We recognize an $\mathcal{L}^2$ norm looking thing. By the Plancharel theorem,

$$\int_{\mathbb{R}} |f \star f'|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f \star f'}|^2 dt.$$

Now we use the theorem on the properties of the Fourier transform which says

$$\widehat{f \star f'}(\xi) = \hat{f}(\xi) \widehat{f'}(\xi).$$

Now we use that same theorem to say that

$$\widehat{f'}(\xi) = i\xi \hat{f}(\xi).$$

So, the stuff on the right is

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi) i\xi \hat{f}(\xi)|^2 d\xi.$$

We are given what the Fourier transform is, so we put it in there:

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\xi^2}{(|\xi|^3+1)^4} d\xi.$$

Now this isn't so terrible. It's an even function so this is

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{\xi^2}{(\xi^3+1)^4} d\xi.$$

It just so happens that the derivative of

$$\frac{1}{(\xi^3+1)^3} \text{ is } \frac{-9\xi^2}{(\xi^3+1)^4},$$

so

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi^2}{(\xi^3+1)^4} d\xi = \frac{-1}{9\pi} \left. \frac{1}{\xi^3+1} \right|_0^\infty = \frac{1}{9\pi}.$$

So, now you know how to use these techniques to do two practical things: (1) solve the IVP for the homogeneous heat equation and (2) compute scary integrals. If you have an inhomogeneous IVP for the heat equation, here are two ways to deal with that:

(1) If the inhomogeneity is *time independent,* look for a steady state solution to solve the inhomogeneous equation. Then, solve the homogeneous equation, but change your initial data. If f is your steady state solution and v was your initial data (before f came along), solve the IVP for the homogeneous heat equation with IC $v - f$ rather than just v.
(2) If the inhomogeneity is *time dependent,* you can try to solve using the original method we did, that is by Fourier transforming the whole PDE.

1.2. **The Sampling Theorem.**

Theorem 2. *Let $f \in L^2(\mathbb{R})$. We take the definition of the Fourier transform of f to be*

$$\int_{\mathbb{R}} e^{-ix\xi} f(x) dx,$$

and we then assume that there is $L > 0$ so that $\hat{f}(\xi) = 0$ $\forall \xi \in \mathbb{R}$ with $|\xi| > L$. Then:

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

Proof: This theorem is all about the interaction between Fourier series and Fourier coefficients and how to work with both simultaneously. Since the Fourier transform $\hat{f}$ has compact support, the following equality holds as elements of $L^2([-L, L])$,

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx.$$

We use the Fourier inversion theorem (FIT) to write

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt} \hat{f}(x) dx = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt} \hat{f}(x) dx.$$

On the left we have used the fact that $\hat{f}$ is supported in the interval $[-L, L]$, thus the integrand is zero outside of this interval, so we can throw that part of the integral away.

We next substitute the Fourier expansion of $\hat{f}$ into this integral,

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} c_n e^{in\pi x/L} dx.$$

Let us take a closer look at the coefficients

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx = \frac{1}{2L}\int_{\mathbb{R}} e^{ix(-n\pi/L)}\hat{f}(x)dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

In the second equality we have used the fact that $\hat{f}(x) = 0$ for $|x| > L$, so by including that part we don't change the integral. In the third equality we have used the FIT!!! So, we now substitute this into our formula above for

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty}\frac{\pi}{L} f\left(\frac{-n\pi}{L}\right)e^{in\pi x/L}dx$$

This is approaching the form we wish to have in the theorem, but the argument of the function f has a pesky negative sign. That can be remedied by switching the order of summation, which does not change the sum, so

$$f(t) = \frac{1}{2L}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)e^{-in\pi x/L}dx.$$

We may also interchange the summation with the integral[1]

$$f(t) = \frac{1}{2L}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\int_{-L}^{L} e^{x(it-in\pi/L)}dx.$$

We then compute

$$\int_{-L}^{L} e^{x(it-in\pi/L)}dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)}\sin(Lt-n\pi).$$

Substituting,

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\frac{\sin(Lt-n\pi)}{Lt-n\pi}.$$

Of course my dyslexia has ended up with things being backwards, but it is not a problem because sine is odd so

$$\sin(Lt-n\pi) = -\sin(n\pi-Lt),$$

so

$$\frac{\sin(Lt-n\pi)}{Lt-n\pi} = \frac{-\sin(n\pi-Lt)}{Lt-n\pi} = \frac{\sin(n\pi-Lt}{n\pi-Lt}.$$

[1] None of this makes sense pointwise; we are working over L^2. The key property which allows interchange of limits, integrals, sums, derivatives, etc is *absolute convergence.* This is the case here because elements of L^2 have $\int |f|^2 < \infty$. That is precisely the type of absolute convergence required.