# FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Let's do an example. Say we want to solve an inhomogeneous heat equation on $\mathbb{R}$. So, we are solving

$$u_t - u_{xx} = G(x,t), \quad u(x,0) = v(x).$$

Let's try the Fourier transform method:

$$\partial_t \hat{u}(\xi,t) + \xi^2 \hat{u}(\xi,t) = \hat{G}(\xi,t).$$

This is a first order ODE. If you are a CHEMIST, then you did the special extra part of the course and actually learned how to solve this ODE in $t$. Pretty cool. To see how this works, treat $\xi$ like a constant, and write

$$f'(t) + \xi^2 f(t) = \hat{G}(\xi,t).$$

The method says to first compute

$$e^{\int \xi^2 dt} = e^{\xi^2 t}.$$

Next compute

$$\int e^{\xi^2 t} \hat{G}(\xi,t) dt.$$

Then, the solution is

$$\frac{\int e^{\xi^2 t} \hat{G}(\xi,t) dt + C(\xi)}{e^{\xi^2 t}} = e^{-\xi^2 t} \int e^{\xi^2 s} \hat{G}(\xi,s) ds + C(\xi) e^{-\xi^2 t}.$$

Now, if we choose a primitive (anti-derivative) of $e^{\xi^2 s} \hat{G}(\xi,s)$ which is zero when $t=0$, then we can simply set $C(\xi) = \hat{v}(\xi)$. So, to do this, we use the function

$$F(\xi,t) := \int_0^t e^{\xi^2 s} \hat{G}(\xi,s) ds.$$

By the Fundamental Theorem of Calculus, the $t$ derivative of $F$ is the integrand evaluated at $t$. There is too much $t$. Let me be more precise

$$\partial_t F(\xi,t)|_{t=t_0} = e^{\xi^2 t_0} \hat{G}(\xi,t_0).$$

That's what the FTC says. So, our solution as of now looks like

$$\hat{u}(\xi, t) = e^{-\xi^2 t} \int_0^t e^{\xi^2 s} \hat{G}(\xi, s) ds + \hat{v}(\xi) e^{-\xi^2 t}.$$

We need to figure out from whence this Fourier transform came (equivalently, invert the Fourier transform). This is a linear process, so we can deal with each piece separately and then add them. Well, the second part we did last time. We saw that the Fourier transform of

$$\frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y) dy$$

is

$$\hat{v}(\xi) e^{-\xi^2 t}.$$

Similarly, let's look at the first part. It is

$$e^{-\xi^2 t} \int_0^t e^{\xi^2 s} \hat{G}(\xi, s) ds = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi, s) ds.$$

By the same calculations, the Fourier transform of

$$\frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} G(y, s) dy = e^{-\xi^2 (t-s)} \hat{G}(\xi, s).$$

Yet again playing switch-a-roo with limits[1],

$$\mathcal{F}\left( \int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} G(y, s) dy ds \right)(\xi) = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi, s) ds.$$

Therefore, our full solution is

$$\int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} G(y, s) dy ds + \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y) dy.$$

1.1. **Fourier sine and cosine transform.** Let's start with a motivating example. We want to solve the heat equation on a half line. Imagine we have a giant rod which is say insulated at the one end and goes out to infinity at the other. It has some initial temperature distribution given by a function $f(x)$. So, we are solving

$$u_t - u_{xx} = 0, \quad u_x(0, t) = 0, \quad u(x, 0) = f(x), \quad x \in [0, \infty).$$

Let's start with the initial data. It is only defined on a half line. So, to use Fourier transform techniques let us consider extending it evenly or oddly. Denote these by $f_e$ and $f_o$ respectively. When we compute the Fourier transform:

$$\hat{f}_e(\xi) = \int_{\mathbb{R}} f_e(x) e^{-ix\xi} dx = \int_{\mathbb{R}} f_e(x)(\cos(x\xi) - i \sin(x\xi)) dx = 2 \int_0^\infty f(x) \cos(x\xi) dx.$$

On the other hand

$$\hat{f}_o(\xi) = \int_{\mathbb{R}} f_o(x) e^{-ix\xi} dx = \int_{\mathbb{R}} f_o(x)(\cos(x\xi) - i \sin(x\xi)) dx = -2i \int_0^\infty f(x) \sin(x\xi) dx.$$

In this way we arrive at

[1] Trust me!

**Definition 1.** Let $f$ be in $\mathcal{L}^1$ or $\mathcal{L}^2$ on $(0,\infty)$. The Fourier cosine transform,

$$\mathcal{F}_c(f)(\xi) := \int_0^\infty f(x)\cos(\xi x)dx.$$

The Fourier sine transform,

$$\mathcal{F}_s(f)(\xi) := \int_0^\infty f(x)\sin(\xi x)dx.$$

Look at

$$\hat{f}_e(\xi) = 2\int_0^\infty f(x)\cos(x\xi)dx = 2\mathcal{F}_c(f)(\xi).$$

Therefore,

$$\frac{1}{2}\hat{f}_e(\xi) = \mathcal{F}_c(f)(\xi) \implies \hat{f}_e(\xi) = 2\mathcal{F}_c(f)(\xi).$$

Since cosine is even,

$$\hat{f}_e(\xi) = \hat{f}_e(-\xi).$$

The inversion formula (FIT) says that

$$f_e(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}_e(\xi)d\xi = \frac{1}{\pi}\int_{\mathbb{R}} e^{ix\xi}\mathcal{F}_c(f)(\xi)d\xi.$$

Since $\mathcal{F}_c(f)$ is an even function, using the evenness and oddness of cosine and sine, we see that

$$f_e(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(f)(\xi)\cos(x\xi)d\xi.$$

This is the Fourier cosine inversion formula!

Let us repeat this calculation for the odd extension. We compute

$$\hat{f}_o(\xi) = \int_{\mathbb{R}} f_o(x)e^{-ix\xi}dx = \int_{\mathbb{R}} f_o(x)(\cos(x\xi)-i\sin(x\xi))dx = -2i\int_0^\infty f(x)\sin(x\xi)dx.$$

Therefore,

$$\hat{f}_o(\xi) = -2i\mathcal{F}_s(f)(\xi) \implies \frac{i}{2}\hat{f}_o(\xi) = \mathcal{F}_s(f)(\xi).$$

Note that we also see from the oddness of sine that $\mathcal{F}_s(f)(\xi)$ is odd. The standard FIT says that

$$f_o(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}_o(\xi)d\xi = -\frac{i}{\pi}\int_{\mathbb{R}} e^{ix\xi}\mathcal{F}_s(f)(\xi)d\xi.$$

By the evenness of cosine and oddness of sine, this is

$$-\frac{2i}{\pi}\int_0^\infty i\sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi.$$

So, we see that

$$f_o(x) = \frac{2}{\pi}\int_0^\infty \sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi.$$

This is the Fourier sine inversion formula!

Let's return now to our motivating example. Imagine we have solved the problem, so we have a solution $u(x,t)$ defined on $[0,\infty)_x \times [0,\infty)_t$. If we extend evenly, we would get

$$u_e(x,t) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(u)(\xi)\cos(x\xi)d\xi.$$

If we extend oddly, we would get

$$u_o(x,t) = \frac{2}{\pi} \int_0^\infty \sin(x\xi) \mathcal{F}_s(f)(\xi) d\xi.$$

Of course, since these are extensions we are getting our original $u$ function if $x \in [0, \infty)$. To see which one we want to use we LOOK AT THE BOUNDARY CONDITION. We need the derivative with respect to $x$ to vanish at $x = 0$. So, throwing convergence concerns to the wind, we differentiate under the integral[2]

$$\partial_x u_e(x,t) = -\frac{2}{\pi} \int_0^\infty \mathcal{F}_c(u)(\xi) \xi \sin(x\xi) d\xi \implies \partial_x u_e(0,t) = 0.$$

On the other hand

$$\partial_x u_o(x,t) = \frac{2}{\pi} \int_0^\infty \xi \cos(x\xi) \mathcal{F}_s(u)(\xi) d\xi \implies \partial_x u_o(0,t) = \frac{2}{\pi} \int_0^\infty \xi \mathcal{F}_s(u)(\xi) d\xi = ???$$

The even extension automatically gives us the desired boundary condition whereas the odd extension leads to something complicated looking, which we have no reason to know is zero. So, let's stick with the even extension and see if it's going to work...

So, let's proceed like this: take our initial data and extend it evenly to $\mathbb{R}$. Solve the heat equation using the Fourier transform on $\mathbb{R}$ like we did in the previous lecture. We shall get that the solution is

$$\frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f_e(y) e^{-\frac{(x-y)^2}{4t}} dy.$$

Now, let us use the fact that we extended evenly to write this as

$$\frac{1}{2\sqrt{\pi t}} \int_0^\infty f(y) \left( e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

Check that this is an even function:

$$e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} = e^{-\frac{(-x-y)^2}{4t}} + e^{-\frac{(-x+y)^2}{4t}}.$$

AWESOME! So, this means that our solution to the heat equation in this way is EVEN. Therefore, it is the same on the left and right sides. So, we can simply let

$$u(x,t) = u_e(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty f(y) \left( e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

1.2. **Discrete and fast Fourier transform.** We have seen that computing the Fourier transform is not the easiest thing in the world. The example with the Gaussian involving all those tricks: completing the square, complex analysis and contour integral is a nice and easy case. However, in the *real world* you may come across functions and not know how to compute the Fourier transform by hand, nor be able to find it in BETA. It could be lurking in one of our giant handbooks of calculations (Abramowitz & Stegun, Gradshteyn & Rhizik, to name a few). Or it could simply never have been computed analytically. In this case you may compute something called the *discrete Fourier transform.*

[2] I know, I am sounding like a certain orange political "leader" when I say *trust me,* this is really rigorously justifiable. But trust me, it is!

We start with a function, $f(t)$, and think of analyzing $f(t)$ as *time analysis,* whereas analyzing $\hat{f}(\xi)$ as *frequency analysis.* We shall consider a finite dimensional Hilbert space:

$$\mathbb{C}^N = \left\{ (s_n)_{n=0}^{N-1}, \quad s_n \in \mathbb{C}, \quad \langle (s_n), (t_n) \rangle := \sum_{n=0}^{N-1} s_n \overline{t_n} \right\}.$$

Now let

$$e_k(n) := \frac{e^{2\pi i k n/N}}{\sqrt{N}}.$$

**Proposition 2.** *Let*

$$e_k := (e_k(n))_{n=0}^{N-1}.$$

*Then*

$$\{e_k\}_{k=0}^{N-1}$$

*are an ONB of* $\mathbb{C}^N$*.*

**Proof:** We simply compute. It is so cute and discrete!

$$\langle e_k, e_j \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k n/N} e^{-2\pi i j n/N} = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i (k-j) n/N}.$$

If $j = k$ the terms are all 1, and so the total is $N$ which divided by $N$ gives 1. Otherwise, we may without loss of generality assume that $k > j$ (swap names if not the case). Then we are staring at a geometric series! We know how to sum it

$$\sum_{n=0}^{N-1} e^{2\pi i (k-j) n/N} = \frac{1 - e^{2\pi i (k-j) N/N}}{1 - e^{2\pi i (k-j)/N}} = 0.$$

Here it is super important that $k-j$ is a number between 1 and $N-1$. We know this because $0 \le j < k \le N-1$. Hence when we subtract $j$ from $k$, we get something between 1 and $N-1$. So we are not dividing by zero!

Now we shall fix $T$ small and $N$ large and look at $f(t)$ just on the interval $[0, (N-1)T]$. Let

$$f(n) := f(t_n) := f(nT), \quad t_n = nT.$$

Basically, we're going to identify $f$ with an element of $\mathbb{C}^N$, namely

$$(f(n))_{n=0}^{N-1}.$$

**Definition 3.** The discrete Fourier transform is for

$$w_k := \frac{2\pi k}{NT}$$

defined to be

$$F_k = F(w_k) := \langle (f(n)), e_k \rangle = \sum_{n=0}^{N-1} \frac{f(t_n) e^{-2\pi i k n/N}}{\sqrt{N}}.$$

This can also be written as

$$\sum_{n=0}^{N-1} \frac{f(t_n) e^{-i w_k t_n}}{\sqrt{N}}.$$

**Proposition 4.** *We have the inversion formula*

$$f(t_n) = \sum_{k=0}^{N-1} F(w_k) e_n(k) = \langle (F_k), \overline{e}_n \rangle.$$

**Proof:** We simply compute this stuff. By definition

$$\langle (F_k), \overline{e}_n \rangle = \sum_{k=0}^{N-1} F(w_k) e_n(k),$$

because taking two conjugates gives us back the original guy. Now, we insert the definition of $F(w_k)$ which gives us another sum, so we use a different index there. Hence we have

$$\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \frac{f(t_m) e^{-iw_k t_m}}{\sqrt{N}} \frac{e^{2\pi i k n/N}}{\sqrt{N}} = \frac{1}{N} \sum \sum f(t_m) e^{-2\pi i k m/N} e^{2\pi i k n/N}$$

$$= \frac{1}{N} \sum \sum f(t_m) e^{2\pi i k(n-m)/N} = \frac{1}{N} \sum_{m=0}^{N-1} f(t_m) \sum_{k=0}^{N-1} e^{2\pi i k(n-m)/N}$$

$$= \sum_{m=0}^{N-1} f(t_m) \sum_{k=0}^{N-1} \frac{e^{-2\pi i k m/N}}{\sqrt{N}} \frac{\overline{e^{-2\pi i k n/N}}}{\sqrt{N}} = \sum_{m=0}^{N-1} f(t_m) \langle e_m, e_n \rangle.$$

By the proposition we just proved before,

$$\langle e_m, e_n \rangle = \delta_{n,m} = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases}$$

So, the only term which survives is when $m = n$, and so we get

$$f(t_n).$$

Now, we can see this as a sort of matrix multiplication. To compute the full frequency Fourier transform vector, we should compute

$$\begin{bmatrix} F(w_0) \\ F(w_1) \\ \dots \\ F(w_{N-1}) \end{bmatrix}.$$

This is given by the product of the matrix

$$\begin{bmatrix} \bar{e}_0 & \bar{e}_1 & \dots \bar{e}_{N-1} \end{bmatrix}$$

whose columns are

$$\bar{e}_n = \begin{bmatrix} e^0 \\ e^{-2\pi i n/N} \\ e^{-2\pi i(2)n/N} \\ \dots e^{-2\pi i k n/N} \\ \dots \\ e^{-2\pi i n(N-1)/N} \end{bmatrix}$$

together with the vector

$$\begin{bmatrix} f(t_0) \\ f(t_1) \\ \dots \\ f(t_{N-1}) \end{bmatrix}$$

That is

$$\begin{bmatrix} F(w_0) \\ F(w_1) \\ \dots \\ F(w_{N-1}) \end{bmatrix} = \begin{bmatrix} \bar{e}_0 & \bar{e}_1 & \dots \bar{e}_{N-1} \end{bmatrix} \begin{bmatrix} f(t_0) \\ f(t_1) \\ \dots \\ f(t_{N-1}) \end{bmatrix}$$

This is a LOT of calculations. We can speed it up by being clever. Many calculations are repeated in fact. Assume that $N = 2^X$ for some giant power $X$. The idea is to split up into even and odd terms. We do this:

$$F(w_k) = \frac{1}{\sqrt{N}} \left[ \sum_{j=0}^{\frac{N}{2}-1} f(t_{2j}) e^{-2\pi i k(2j)/N} + \sum_{j=0}^{\frac{N}{2}-1} f(t_{2j+1}) e^{-2\pi i k(2j+1)/N} \right].$$

We introduce the slightly cumbersome notation:

$$e_N^k(n) = e^{-2\pi i k n/N}.$$

Then,

$$e_N^k(2j) = e^{-2\pi i k(2j)/N} = e^{-2\pi i k j/(N/2)} = e_{N/2}^k(j).$$

Now we only need an $\frac{N}{2} \times \frac{N}{2}$ matrix! You see, writing this way,

$$F(w_k) = \frac{1}{\sqrt{N}} \left[ \sum_{j=0}^{\frac{N}{2}-1} f(t_{2j}) e_{N/2}^k(j) + e_N^k(1) \sum_{j=0}^{\frac{N}{2}-1} f(t_{2j+1}) e_{N/2}^k(j) \right].$$

We can repeat this many times because $N$ is a power of 2. We just keep chopping in half. If we do this as many times as possible, we will need to do on the order of

$$\frac{N}{2} \log_2(N)$$

computations. This is in comparison to the original method which had an $N \times N$ matrix and was thus on the order of $N^2$ computations. For example, if $N = 2^{10}$, then comparing $N^2 = 2^{20}$ to $\frac{N}{2} \log_2 N = 2^9 * 10$, we see that

$$\frac{2^{10} * 5}{2^{20}} = \frac{x}{100} \implies 100 * 2^{10} * 5 = 2^{20} x \implies 2^2 * 5^3 * 2^{10} 2^{-20} = x,$$

so

$$5^3 2^{-8} = x \approx 0.488.$$

This means the amount of work we are doing by using the FFT is less than 0.5% of the work done using the standard DFT. In other words, we save over 99.5% by doing the FFT. That's why it's called FAST.