FOURIER ANALYSIS & METHODS

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2018.02.19

We shall now enter Chapter 8, and learn about another useful transform, known as the Laplace transform.

Definition 1. Assume that

lap0
$$f(t) = 0 \quad \forall t < 0, \tag{1.1}$$

and that there exists $a, C > 0$ such that

lapa
$$|f(t)| \leq Ce^{at} \quad \forall t \geq 0. \tag{1.2}$$

Then for we define for $z \in \mathbb{C}$ with $\Re(z) > a$ the Laplace transform of f at the point z to be

$$\mathfrak{L}f(z) = \hat{f}(-iz) = \int_0^\infty f(t)e^{-zt}dt.$$

Let us verify that this is well defined. To do so, we estimate

$$|\mathfrak{L}f(z)| \leq \int_0^\infty |f(t)e^{-zt}|dt \leq \int_0^\infty Ce^{at}|e^{-zt}|dt = \int_0^\infty e^{at}e^{-\Re(z)t}dt$$
$$= \left. \frac{e^{t(a-\Re(z))}}{a-\Re(z)} \right|_0^\infty = \frac{1}{\Re(z)-a}.$$

Above we have used the fact that

$$\left|e^{\text{complex number}}\right| = e^{\text{real part}}.$$

Due to this beautiful convergence, $\mathfrak{L}f(z)$ is holomorphic in the half plane $\Re(z) > a$. This is because we may differentiate under the integral sign (play limit switcharoo) due to the absolute convergence of the integral (it's that DCT again). Now, the assumption that $f(t) = 0$ for all negative t is not actually necessary, we could just make it so. For this purpose we define the *heavyside* function, commonly denoted by

$$\Theta(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

If we have some f defined on $\mathbb{R}$ which satisfies (1.2) but is not (1.1), we can apply the Laplace transform to Θf. Another thing which can happen is that we have a function which is only defined on $[0, \infty)$. In that case, we can just extend it to be identically zero on $(-\infty, 0)$.

Let's familiarize ourselves with the Laplace transform by demonstrating some of its fundamental properties.

Proposition 2 (Properties of $\mathfrak{L}$)**.** *Assume f and g satisfy (1.2) and (1.1), then*

(1) $\mathfrak{L}f(x+iy) \to 0$ *as* $|y| \to \infty$ *for all* $x > a$.
(2) $\mathfrak{L}f(x+iy) \to 0$ *as* $x \to \infty$ *for all* y.
(3) $\mathfrak{L}(\Theta(t-a)f(t-a))(z) = e^{-az}\mathfrak{L}f(z)$.
(4) $\mathfrak{L}(e^{ct}f(t))(z) = \mathfrak{L}f(z-c)$.
(5) $\mathfrak{L}(f(at)) = a^{-1}\mathfrak{L}f(a^{-1}z)$.
(6) **** If f is continuous and piecewise $\mathcal{C}^1$ on $[0,\infty)$, and f' satisfies (1.2) and (1.1), then*
$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$
(7) $\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}f(z)$.
(8) $\mathfrak{L}(tf(t))(z) = -(\mathfrak{L}f)'(z)$.
(9) $\mathfrak{L}(f * g)(z) = \mathfrak{L}f(z)\mathfrak{L}g(z)$.

Proof: There's a bunch of stars next to #6 because it's the reason the Laplace transform is useful for solving PDEs and ODEs. It's quite similar to how the Fourier transform takes in derivatives and spits out multiplication. Intuitively, this fact about $\mathfrak{L}$ should jive with the similar fact about $\mathcal{F}$ because well, the Laplace transform is just the Fourier transform taken at a complex point.

The first statement
$$\mathfrak{L}f(z) = \int_0^\infty e^{-(x+iy)t}f(t)dt = \int_0^\infty e^{-xt}f(t)e^{-iyt}dt = \hat{g}(y),$$
for the function
$$g(t) = e^{-xt}f(t).$$
The Riemann-Lebesgue lemma says that $\hat{g}(y) \to 0$ when $|y| \to \infty$. The second statement is more satisfying because we just compute and estimate directly. We did this estimate above already, where we got
$$|\mathfrak{L}f(z)| \leq \frac{1}{\Re(z) - a} \to \infty \text{ when } \Re(z) = x \to \infty.$$
The third one is also a direct computation:
$$\mathfrak{L}(\Theta(t-a)f(t-a))(z) = \int_0^\infty \Theta(t-a)f(t-a)e^{-zt}dt = \int_{-a}^\infty \Theta(s)f(s)e^{-z(s+a)}ds.$$
Above we did the substitution $s = t-a$ so $ds = dt$. Since f and the Heavyside function are zero for negative s, and the Heavyside function is 1 for positive s, this is
$$e^{-za}\int_0^\infty f(s)e^{-zs}ds = e^{-za}\mathfrak{L}f(z).$$
Similarly, we directly compute
$$\mathfrak{L}(e^{ct}f)(z) = \int_0^\infty e^{ct}e^{-zt}f(t)dt = \int_0^\infty e^{-(z-c)t}f(t)dt = \mathfrak{L}f(z-c).$$
Again no surprise, we compute
$$\mathfrak{L}(f(at))(z) = \int_0^\infty e^{-zt}f(at)dt = \int_0^\infty e^{-zs/a}f(s)\frac{ds}{a} = a^{-1}\mathfrak{L}f(z/a).$$

Here we used the substitution $s = at$ so $a^{-1}ds = dt$. Now we are finally getting to the important one:

$$\mathfrak{L}(f')(z) = \int_0^\infty e^{-zt} f'(t)dt = e^{-zt} f(t)\big|_0^\infty + \int_0^\infty z e^{-zt} f(t)dt.$$

We have used integration by parts above. By (1.2) and since $\Re(z) > a$, the limit as $t \to \infty$ is zero, and so we get

$$\mathfrak{L}(f')(z) = -f(0) + z\mathfrak{L}f(z).$$

Awesome. Next we define

$$F(t) = \int_0^t f(s)ds.$$

Then, we use the preceding fact:

$$\mathfrak{L}(F')(z) = z\mathfrak{L}F(z) - F(0) = z\mathfrak{L}F(z).$$

Since $F' = f$ we get

$$z^{-1}\mathfrak{L}(f)(z) = \mathfrak{L}(\int_0^t f(s)ds)(z).$$

Next, we compute:

$$\mathfrak{L}(tf(t))(z) = \int_0^\infty t e^{-zt} f(t)dt = \int_0^\infty \frac{d}{dz}\left(-e^{-zt}\right) f(t)dt$$
$$= \frac{d}{dz}\left(-\int_0^\infty e^{-zt} f(t)dt\right) = -(\mathfrak{L}f)'(z).$$

Yes, we have used the absolute convergence of the integral to swap limits. It's legit yo. Finally,

$$\mathfrak{L}(f * g)(z) = \mathcal{F}(f * g)(-iz) = \hat{f}(-iz)\hat{g}(-iz) = \mathfrak{L}f(z)\mathfrak{L}g(z).$$

We may also use the notation

$$\widetilde{f}(z) = \mathfrak{L}f(z).$$

1.1. **Applications to solving PDEs and ODEs.** We see that

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Let's do it again:

$$\mathfrak{L}(f'')(z) = z\mathfrak{L}(f')(z) - f'(0) = z\left(z\mathfrak{L}f(z) - f(0)\right) - f'(0) = z^2\mathfrak{L}f(z) - zf(0) - f'(0).$$

In general:

Proposition 3. *Assume that everything is defined, then*

$$\mathfrak{L}(f^{(k)})(z) = z^k\mathfrak{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0)z^{j-1}.$$

Proof: Well clearly we should do a proof by induction! Check the base case first:

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Here $k = 1$ and the sum has only one term with $j = k = 1$. It works. Now we assume the above formula holds and we show it for $k + 1$. We compute

$$\mathfrak{L}(f^{(k+1)})(z) = \mathfrak{L}((f^{(k)})')(z) = z\mathfrak{L}(f^{(k)})(z) - f^{(k)}(0).$$

By induction this is

$$z\left(z^k\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}\right) - f^{(k)}(0).$$

This is

$$z^{k+1}\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0).$$

Let us change our sum: let $j + 1 = l$. Then our sum is

$$\sum_{l=2}^{k+1} f^{k-(l-1)}(0)z^{l-1} = \sum_{l=2}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

Observe that

$$f^{(k)}(0) = f^{k+1-1}(0)z^{1-1}.$$

Hence

$$-\sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0) = -\sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

So, we have computed

$$\mathfrak{L}(f^{(k+1)})(z) = z^{k+1}\mathfrak{L}f(z) - \sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

That is the formula for $k + 1$, which is what we needed to obtain.

For this reason one can use $\mathfrak{L}$ to solve linear constant coefficient ODEs which can be *non-homogeneous!* Let us see how this works...

1.2. **Solving ODEs.** A linear, constant coefficient ODE of order n looks like:

$$\sum_{k=0}^{n} c_k u^{(k)}(t) = f(t).$$

In order for the solution to be unique, there must be specified initial conditions on u, that is

$$u(0), u'(0), \ldots u^{(n-1)}(0).$$

We are not requiring $f(t)$ to be the zero function, so the ODE could be *inhomogeneous.* Notoriously difficult to solve right? NOT ANYMORE! We hit both sides of the ODE with $\mathfrak{L}$:

$$\sum_{k=0}^{n} c_k \mathfrak{L}(u^{(k)})(z) = \widetilde{f}(z).$$

Let's write out the left side using our proposition. First we have

$$c_0\widetilde{u}(z).$$

Then we have

$$c_1\left(z\widetilde{u}(z) - u(0)\right).$$

In general we have

$$c_k\left(z^k\widetilde{u}(z) - \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}\right).$$

So, if we collect all the terms with $\widetilde{u}(z)$, we get

$$(c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n)\widetilde{u}(z) = P(z)\widetilde{u}(z),$$

$$P(z) = \sum_{k=0}^{n} c_k z^k.$$

Now let's collect all the rest:

$$-\sum_{k=1}^{n} c_k \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1} = Q(z).$$

This is just a polynomial also. So our ODE has been LAPLACE-TRANSFORMED into

$$P(z)\widetilde{u}(z) + Q(z) = \widetilde{f}(z).$$

We can solve this for $\widetilde{u}(z)$:

$$\widetilde{u}(z) = \frac{\widetilde{f}(z) - Q(z)}{P(z)}.$$

Hence to get our solution $u(t)$ we just need to invert the Laplace transform of the right side, that is our solution will be

$$u(t) = \mathfrak{L}^{-1}\left(\frac{\widetilde{f}(z) - Q(z)}{P(z)}\right).$$

1.3. **Solving PDEs.** We look at a general equation known as the *telegraph equation,*

$$u_{xx} = \alpha u_{tt} + \beta u_t + \gamma u.$$

This is homogeneous, and generalizes both the heat equation ($\alpha = \gamma = 0$, and $\beta = 1$) as well as the wave equation ($\beta = \gamma = 0$, and $\alpha = 1$). Apparently it corresponds to an electromagnetic signal on a cable. I'll just trust Folland when he says that. Let's consider this problem on a half line, with the conditions

$$u(0,t) = f(t), \quad u(x,0) = u_t(x,0) = 0.$$

YIKES! Our boundary condition at $x = 0$ is *a function of* t. We haven't see that before! We hit the whole PDE with the Laplace transform *in the* t *variable.* This gives

$$\widetilde{u}_{xx}(x,z) = \alpha\mathfrak{L}(u_{tt})(x,z) + \beta\mathfrak{L}(u_t)(x,z) + \gamma\widetilde{u}(x,z).$$

Next, we use the lovely conditions that $u(x,0) = u_t(x,0) = 0$, together with our proposition, so we have

$$\widetilde{u}_{xx}(x,z) = \alpha z^2\widetilde{u}(x,z) + \beta z\widetilde{u}(x,z) + \gamma\widetilde{u}(x,z).$$

This is simply

$$\widetilde{u}_{xx}(x,z) = \left(\alpha z^2 + \beta z + \gamma\right) \widetilde{u}(x,z).$$

It's a second order, linear, constant coefficient, homogeneous ODE for the x variable. Let

$$q = \sqrt{\alpha z^2 + \beta z + \gamma}.$$

Our solution to the ODE is of the form

$$\widetilde{u}(x,z) = a(z)e^{qx} + b(z)e^{-qx}.$$

We have that lovely BC at $x = 0$: $u(0,t) = f(t)$. Hence,

$$\widetilde{u}(0,z) = \widetilde{f}(z) \implies a(z) + b(z) = \widetilde{f}(z).$$

Note that here we are extending f to $(-\infty, 0)$ to be identically equal to zero so that we may Laplace transform it. Also, we presume that f satisfies (lapa 1.2). Assume that $\Re(q) > 0$. (If this weren't the case, just swap q and $-q$). To be able to invert the Laplace transform and get the solution to our PDE, we will not want $\widetilde{u}(x,z) \to \infty$ when $x \to \infty$. Hence, we throw out the e^{qx} solution and just use

$$\widetilde{u}(x,z) = b(z)e^{-qx}.$$

Therefore, $b(z) = \widetilde{f}(z)$. So, our Laplace-transformed solution is

$$\widetilde{u}(x,z) = \widetilde{f}(z)e^{-qx}.$$

By the properties of the Laplace transform, if we can find $g(x,t)$ such that

$$\widetilde{g}(x,z) = e^{-qx},$$

then the solution to this PDE will be

soln

$$u(x,t) = f * g(x,t) = \int_{\mathbb{R}} f(t-s)g(x,s)ds = \int_0^t f(t-s)g(x,s)ds. \tag{1.3}$$

This is because f is zero for negative times.

As a particular example of this, let's consider the heat equation. Then, we have $\alpha = \gamma = 0$, and $\beta = 1$, so

$$q = \sqrt{z}.$$

So, our Laplace-transformed solution looks like

$$\widetilde{f}(z)e^{-\sqrt{z}x}.$$

Since the Laplace transform turns convolutions into multiplication, we would like to find $g(x,t)$ so that

$$\widetilde{g}(x,z) = e^{-\sqrt{z}x}.$$

Then, the solution will be given as in (soln 1.3).

To find this, we need to invert the Laplace transform. How do we do this? STAY TUNED! If the suspense is too much, I think this is super cute and may provide some comic relief...

`https://www.youtube.com/watch?v=gt6CLLn539o`