**FOURIER ANALYSIS & METHODS**

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Let us recall that

$$\mathfrak{L}f(z) = \int_0^\infty e^{-zt} f(t) dt = \int_0^\infty e^{-\Re(z)t} e^{-i\Im(z)t} f(t) dt.$$

For this to be well defined we assume that $f$ satisfies:

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$$f(t) = 0 \quad \forall t < 0, \tag{1.1}$$

and that there exists $a, C > 0$ such that

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$$|f(t)| \leq Ce^{at} \quad \forall t \geq 0. \tag{1.2}$$

We begin with one more property of the Laplace transform.

**Proposition 1.** *If $t^{-1}f(t)$ satisfies (1.1) and (1.2), then*

$$\mathfrak{L}(t^{-1}f(t))(z) = \int_z^\infty \mathfrak{L}f(w) dw.$$

*The integral is any contour in the $w$-plane which starts at $z$ along which $\Im w$ stays bounded and $\Re w \to \infty$.*

**Proof:** Note that by (1.2), if $t^{-1}f(t)$ satisfies this, then at the point $t = 0$ apparently the function $f$ vanishes, so that the function $t^{-1}f(t)$ is well defined. So, don't panic about this point!!! We next define the holomorphic function

$$F(z) = \int_z^\infty \widetilde{f}(w) dw.$$

Since $\widetilde{f}(w) \to 0$ when $\Re(w) \to \infty$ and $\Im(w)$ stays bounded, the fundamental theorem of calculus says that

$$F'(z) = -\widetilde{f}(z).$$

On the other hand,

$$\frac{d}{dz} \int_0^\infty t^{-1} f(t) e^{-zt} dt = \int_0^\infty -f(t) e^{-zt} dt = -\widetilde{f}(z).$$

Hence,

$$F(z) = \int_0^\infty t^{-1} f(t) e^{-zt} dt + c,$$

for some constant $c$. Since

$$\lim_{\Re z \to \infty} F(z) = 0 = \lim_{\Re(z) \to \infty} \int_0^\infty t^{-1} f(t) e^{-zt} dt \implies c = 0.$$

1.1. **Inverting the Laplace transform.** We know that the Laplace transform is closely related to the Fourier transform. Let's write it down

$$\widetilde{f}(z) = \int_0^\infty f(t) e^{-zt} dt = \int_0^\infty f(t) e^{-\Re(z)t - i\Im(z)t} dt.$$

For this reason, let's define

$$g(t) = e^{-\Re(z)t} f(t),$$

so we also have

$$f(t) = e^{\Re(z)t} g(t).$$

Then

$$\mathfrak{L}f(z) = \hat{g}(\Im(z)) = \int_{\mathbb{R}} f(t) e^{-\Re(z)} e^{-i\Im(z)t} dt,$$

because $f(t) = 0$ for all $t < 0$. Let's apply the FIT to the function, $g$:

$$g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi) e^{i\xi t} d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \mathfrak{L}f(\Re(z) + i\xi) e^{i\xi t} d\xi.$$

To make this look less imposing, let us write $y = \xi$. So, we have

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \widetilde{f}(\Re(z) + iy) e^{iyt} dy.$$

Since $f(t) = e^{\Re(z)t} g(t)$, we have

$$f(t) = e^{\Re(z)t} \frac{1}{2\pi} \int_{-\infty}^\infty \widetilde{f}(\Re(z) + iy) e^{iyt} dy = \frac{1}{2\pi} \int_{-\infty}^\infty \widetilde{f}(\Re(z) + iy) e^{\Re(z)t + iyt} dy.$$

We would like to understand this as a complex integral. If we parametrize the vertical path for $w \in \mathbb{C}$ with $\Re(w) = \Re(z)$ by $w = \Re(z) + iy$, then $dw = idy$. We do not have an $i$. Hence, what we are computing is

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma_z} \widetilde{f}(w) e^{wt} dw,$$

where $\Gamma_z$ is the upward vertical path along the line $\Re(w) = \Re(z)$. This is the LIT: Laplace inversion formula:

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma_z} \widetilde{f}(w) e^{wt} dw.$$

By definition of the Laplace transform, this should hold for $z \in \mathbb{C}$ with $\Re(z) > a$ where $a$ comes from (1.2). If we naively look at this equation, we see that the left side is *independent of $z$*. So, the right side ought to be as well.

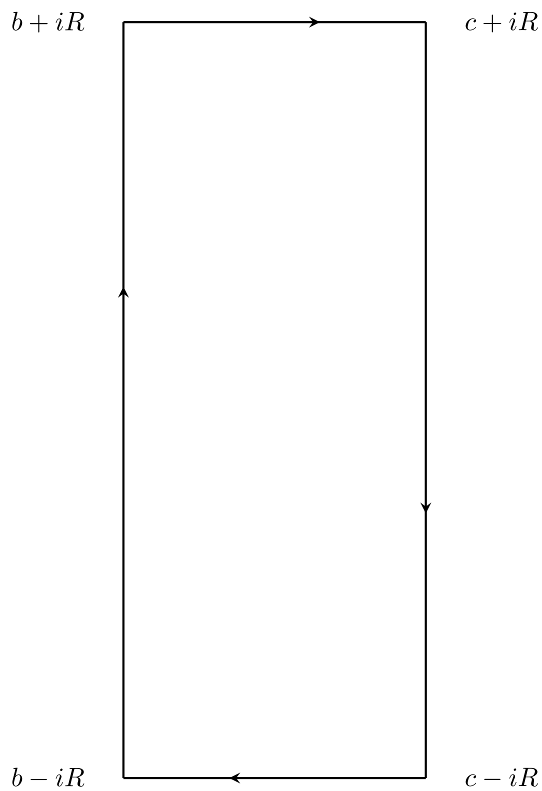


FIGURE 1. The contour over which we integral. Call the contour $\Gamma_R$. As one can see, we assume that $c > b$.

box

**Theorem 2** (LIT). *Let $F(z)$ be analytic in $\Re(z) > a$. For $b > a$, $R > 0$, and $t \in \mathbb{R}$, let*

$$f_{R,b}(t) = \frac{1}{2\pi i}\int_{b-iR}^{b+iR} F(z)e^{zt}dz.$$

*Assume that for some $\alpha > 1/2$ and $C > 0$ we have*

$$|F(z)| \leq C(1+|z|)^{-\alpha}, \quad \forall z \in \mathbb{C} \text{ with } \Re(z) > a,$$

*and assume that for some $b > a$, $f_{R,b}(t)$ converges pointwise as $R \to \infty$ to some $f(t)$ which satisfies (1.1) and (1.2). Then*

$$\lim_{R\to\infty} f_{R,b}(t) = f(t) \quad \forall b > a,$$

*and*

$$F(z) = \mathfrak{L}f(z).$$

**Proof:** Let us draw and define a contour, with our amazing tikz skillz yo.

By assumption the function $F$ is analytic inside the box, and $e^{zt}$ is an entire function. Therefore

$$\int_{\Gamma_R} F(z)e^{zt}dz = 0.$$

So, we wish to show that the limit as $R \to \infty$ of the top and bottom integrals is zero. To obtain this, we either wave our hands like Folland or actually estimate:

$$\int_{b \pm iR}^{c \pm iR} |F(z)||e^{zt}| dz \le |c-b| e^{ct} \max_{b \le x \le c} \frac{C}{(1+|x \pm iR|)^\alpha}.$$

Above we used the fact that between $b \pm iR$ and $c \pm iR$, $|e^{zt}| \le e^{ct}$ together with the estimate assumed on $F$. Next, we note that

$$|x \pm iR| = \sqrt{x^2 + R^2} \ge R.$$

Therefore we estimate from above by

$$|c-b| e^{ct} \frac{C}{(1+R)^\alpha} \to 0 \text{ as } R \to \infty.$$

Therefore, if for some $b > a$,

$$\lim_{R \to \infty} f_{R,b}(t) = f(t),$$

this means that

$$\lim_{R \to \infty} \int_{b-iR}^{b+iR} F(z) e^{zt} dz - \int_{c-iR}^{c+iR} F(z) e^{zt} dz = 0.$$

To see this, observe that

$$\int_{\Gamma_R} F(z) e^{zt} dz = 0 \quad \forall R.$$

Moreover, the top and bottom integrals go to zero as $R \to \infty$. Hence the sum of the left and right integrals also tends to zero as $R \to \infty$. So,

$$\lim_{R \to \infty} \int_{b-iR}^{b+iR} F(z) e^{zt} dz = \lim_{R \to \infty} \int_{c-iR}^{c+iR} F(z) e^{zt} dz \implies \lim_{R \to \infty} f_{R,b}(t) = f(t) = \lim_{R \to \infty} f_{R,c}(t).$$

Now, let us parametrize the complex integral. We use $\gamma(s) = b + is$ so $\dot{\gamma}(s) = i ds$. Hence

$$\int_{b-iR}^{b+iR} F(z) e^{zt} dz = \int_{-R}^{R} F(b+is) e^{(b+is)t} i ds = i e^{bt} \int_{-R}^{R} F(b+is) e^{ist} ds.$$

Moreover, we have assumed that

$$\lim_{R \to \infty} f_{R,b}(t) = \lim_{R \to \infty} \frac{i e^{bt}}{2\pi i} \int_{-R}^{R} F(b+is) e^{ist} ds = f(t)$$

so

$$\lim_{R \to \infty} \int_{-R}^{R} F(b+is) e^{ist} ds = 2\pi e^{-bt} f(t).$$

Let us define here

$$g_{R,b}(s) = \begin{cases} F(b+is) & |s| \le R \\ 0 & |s| > R \end{cases}.$$

Then

$$\int_{-R}^{R} F(b+is) e^{ist} ds = \int_{\mathbb{R}} g_{R,b}(s) e^{ist} ds = \widehat{g_{R,b}}(-t).$$

Moreover,

$$\lim_{R \to \infty} \widehat{g_{R,b}(-t)} = 2\pi e^{-bt} f(t).$$

Similarly

$$\lim_{R\to\infty} \widehat{g_{R,b}(t)} = 2\pi e^{bt} f(-t).$$

On the other hand,

$$\lim_{R\to\infty} g_{R,b}(s) = F(b+is).$$

By the FIT,

$$F(b+it) = \frac{1}{2\pi}\int_{\mathbb{R}} 2\pi e^{bs} f(-s) e^{its} ds.$$

It is more natural to do a change of variables, letting $\sigma = -s$, so $d\sigma = -ds$, and we get

$$F(b+it) = \int_{\sigma=\infty}^{\sigma=-\infty} e^{-b\sigma} f(\sigma) e^{-it\sigma}(-d\sigma) = \int_{-\infty}^{\infty} e^{-\sigma(b+it)} f(\sigma) d\sigma$$

$$= \int_0^{\infty} e^{-\sigma(b+it)} f(\sigma) d\sigma = \mathfrak{L}f(b+it).$$

Here we use the fact that $f$ satisfies (1.1).

1.2. **Computing an inverse Laplace transform.** For the case in which our telegraph equation is the heat equation, we have $\alpha = \gamma = 0$, and $\beta = 1$, so

$$q = \sqrt{z}.$$

So, our Laplace-transformed solution looks like

$$\widetilde{f}(z) e^{-\sqrt{z}x}.$$

We are therefore looking for $g(x,t)$ so that

$$\widetilde{g}(x,z) = e^{-\sqrt{z}x}.$$

Now, we know from solving the heat equation on $\mathbb{R}$ that we used

$$e^{-x^2/(4t)}(4\pi t)^{-1/2}.$$

So, maybe because the Laplace and Fourier transforms are closely related, we can use this. The idea is to compute its Laplace transform. This probably won't give us the function we want, but maybe the process will show us how to modify the function above in order to get $g(x,t)$ whose Laplace transform is $\widetilde{g}(x,z) = e^{-\sqrt{z}x}$. We proceed like this because the inverse Laplace transform looks pretty scary to compute. So, let us call

$$\star = \int_0^{\infty} e^{-tz} e^{-x^2/(4t)} (4\pi t)^{-1/2} dt.$$

We are computing the Laplace transform of $\Theta(t)h(x,t)$ where

$$h(x,t) = e^{-x^2/(4t)}(4\pi t)^{-1/2}.$$

Now, we see that

$$\star = \int_0^{\infty} (4\pi t)^{-1/2} \exp\left(-(\sqrt{tz})^2 - \left(\frac{x}{2\sqrt{t}}\right)^2\right) dt.$$

We do the completing the square trick in the exponent:

$$\star = \int_0^\infty (4\pi t)^{-1/2} \exp\left(-\left(\sqrt{tz} - \frac{x}{2\sqrt{t}}\right)^2 - x\sqrt{z}\right) dt$$

$$= e^{-x\sqrt{z}} \int_0^\infty \frac{1}{2\sqrt{\pi t}} \exp\left(-\left(\sqrt{tz} - \frac{x}{2\sqrt{t}}\right)^2\right).$$

To compute this we need to use a very very clever trick. First, let us clean up our integral just a little bit to remove that pesky $\sqrt{t}$ which is getting divided. We let $s = \sqrt{t}$. Then

$$ds = \frac{dt}{2\sqrt{t}}$$

So,

$$\star = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi}} \int_0^\infty e^{-(s\sqrt{z} - x/(2s))^2} ds.$$

**Trick 1** (Cauchy-Schlömilch transform)**.**

$$\int_0^\infty af((as - b/s)^2)ds = \int_0^\infty f(y^2)dy.$$

**Proof:** The proof is so clever. I don't know if Cauchy and Schlömilch actually had anything to do with this formula... As a funny aside, Oscar Schlömilch was elected a foreign member of the Royal Swedish Academy of Sciences in 1862. He was a German mathematician who lived 13 April 1823 to 7 February 1901. On the other hand, Cauchy was a French mathematician and physicist who lived 21 August 1789 to 23 May 1857. So, they briefly had some overlap. Did they ever meet? Why is this named after them? It is a big mystery...

We do a substitution in the integral. Let $t = \frac{b}{as}$. Then

$$dt = -\frac{b}{as^2} ds \implies -\frac{as^2}{b} dt = ds.$$

We see that

$$t^2 = \frac{b^2}{a^2 s^2} \implies \frac{a^2 s^2}{b^2} = t^{-2} \implies \frac{as^2}{b} = \frac{b}{at^2}.$$

Next, when $s \to 0$ and $s > 0$ we see that $t \to \infty$. On the other hand, when $s \to \infty$, $t \to 0$. We also see that

$$as = \frac{t}{b}, \quad -\frac{b}{s} = -ta.$$

So, let us call

$$\heartsuit = \int_0^\infty af((as - b/s)^2)ds = \int_\infty^0 af((t/b - ta)^2)\left(-\frac{b}{at^2}\right) dt$$

$$= \int_0^\infty f((t/b - at)^2)\frac{b}{t^2} dt.$$

Note that

$$(t/b - at)^2 = (-(at - t/b))^2 = (at - t/b)^2.$$

Hence we have computed

$$\heartsuit = \int_0^\infty f((at - t/b)^2)\frac{b}{t^2} dt.$$

Therefore

$$2\heartsuit = \int_0^\infty af((as-b/s)^2)ds + \int_0^\infty f((at-t/b)^2)\frac{b}{t^2}dt$$
$$= a\int_0^\infty f((as-b/s)^2)ds + b\int_0^\infty f((as-b/s)^2)\frac{ds}{s^2}.$$

As a consequence,

$$\heartsuit = \frac{1}{2}\int_0^\infty f((as-b/s)^2)\left(a+\frac{b}{s^2}\right)ds.$$

Now we let

$$y = as - \frac{b}{s} \implies dy = a + \frac{b}{s^2}.$$

We note that when $s \to 0$, $y \to -\infty$, and on the flip side, when $s \to \infty$, $y \to \infty$. Therefore

$$\heartsuit = \frac{1}{2}\int_{-\infty}^\infty f(y^2)dy = \int_0^\infty f(y^2)dy,$$

since $f(y^2)$ is an even function.

We will use this transform with

$$a = \sqrt{z}, \quad b = \frac{x}{2}, \quad f(s) = e^{-s^2}.$$

Then, it says that

$$\int_0^\infty \sqrt{z}\exp(-(as-b/s)^2)ds = \int_0^\infty \sqrt{z}\exp\left(-\left(s\sqrt{z}-\frac{x}{2s}\right)^2\right)ds$$
$$= \int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}.$$

Now we were computing

$$\star = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi}}\int_0^\infty e^{-(s\sqrt{z}-x/(2s))^2}ds = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi z}}\int_0^\infty \sqrt{z}e^{-(s\sqrt{z}-x/(2s))^2}ds$$
$$= \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

So, we have computed

$$\mathfrak{L}\left(\Theta(t)h(x,t)\right)(z) = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

We are off by the denominator. However, let us consider

$$\int_z^\infty \frac{e^{-x\sqrt{w}}}{2\sqrt{w}}dw = -\frac{e^{-x\sqrt{w}}}{x}\Bigg|_{w=z}^\infty = \frac{e^{-x\sqrt{z}}}{x}.$$

The property we proved today says

$$\mathfrak{L}(t^{-1}f(t))(z) = \int_z^\infty \widetilde{f}(w)dw.$$

So,

$$\mathfrak{L}(t^{-1}\Theta(t)h(x,t))(z) = \int_z^\infty \frac{e^{-x\sqrt{w}}}{2\sqrt{w}}dw = \frac{e^{-x\sqrt{z}}}{x}.$$

because we computed

$$\mathfrak{L}\left(\Theta(t)h(x,t)\right)(z) = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

We can simply multiply both sides by $x$ to get

$$\mathfrak{L}(t^{-1}x\Theta(t)h(x,t))(z) = e^{-x\sqrt{z}}$$

as desired. Therefore going back to our problem, the solution

$$u(x,t) = (f(s) * (s^{-1}x\Theta(s)h(x,s))(t) = \int_{\mathbb{R}} f(t-s)g(x,s)ds$$
$$= \int_0^t f(t-s)s^{-1}xe^{-s^2/(4s)}(4\pi s)^{-1/2}ds.$$

This is because $f$ is zero for negative times. Here, the function $g$ was the function whose Laplace transform gives us $e^{-x\sqrt{z}}$. We can re-write things a little prettier:

$$u(x,t) = \int_0^t f(t-s)\frac{e^{-x^2/(4s)}x}{s\sqrt{4\pi s}}ds.$$

*Remark* 1. One of the things I love about this class is that you begin to approach actual research mathematics. I think that must be exciting for you, because calculus (envariabelanalys) is like 300 years old. Cauchy's complex analysis is also a few hundred years old. That's not very close to actual current year 2018 math! Here is an example of how the Cauchy-Schlömilch transform is super awesome (and look, this paper is only 8 years old which is super young by research terms):

`https://arxiv.org/abs/1004.2445`