FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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We are on the home stretch! So far, the geometric settings we can handle are:

(1) finite intervals and rectangles, using Fourier series and SLP techniques;
(2) the entire real line, using Fourier transform;
(3) with nice boundary conditions, a half line using Fourier sine/cosine transform;
(4) with a time-dependent boundary condition, a half line using Laplace transform.

1.1. **Magical Bessel functions.** We shall now expand the techniques which worked on finite intervals and rectangles to other geometries. The idea is that the geometry is rectangular if we change coordinates. For example, let's look at a circular sector of radius ρ and opening angle α. In the eyes of polar coordinates, this is a rectangle, $[0, \rho] \times [0, \alpha]$. That is, this set in $\mathbb{R}^2$ is in polar coordinates

$$\{(r, \theta) \in \mathbb{R}^2 : 0 \le r \le \rho, \text{ and } 0 \le \theta \le \alpha\}.$$

This is much the same as how we describe a rectangle using *rectangular* coordinates, (x, y). To solve both the heat equation and the wave equation in a circular sector, we can use the same SLP and Fourier series style techniques we used on rectangles. The heat equation (homogeneous) demands:

$$\partial_t u + \Delta u = 0, \quad \Delta = -\partial_{xx} - \partial_{yy}.$$

The homogeneous wave equation demands:

$$u_{tt} + \Delta u = 0.$$

It's the same Laplace operator, Δ, in both places. If we have neat and tidy (self-adjoint) boundary conditions, we can use separation of variables. Writing our function as $T(t)S(x, y)$, in both cases we need to solve an equation of the form

$$\Delta S = \lambda S.$$

After we solve this, we can then determine T. So, we're going to try to do this but in the geometric setting of a circular sector. Finally, you get some interesting geometry!!

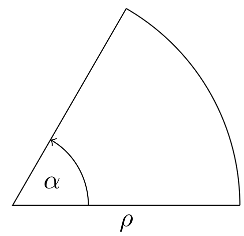

FIGURE 1. A circular sector of opening angle α and radius ρ.

Let's assume that we have the Dirichlet boundary condition on the boundary of the circular sector. So, we are looking for a function S which is zero on the boundary. To formulate this in terms of the rectangular coordinates, we would need to define the boundary of the circular sector using rectangular coordinates. If we do this:

$$x^2 + y^2 = r^2, \quad \text{with } \arctan(y/x) \in [0, \alpha] \text{ is the curved part of the boundary,}$$

and

$$x^2 + y^2 \le r^2, \quad \text{with } \arctan(y/x) = 0 \text{ or } \alpha \text{ are the straight edges.}$$

Defining the boundary this way is SUPER COMPLICATED. AUUUUGGGGGH! Now, on the other hands, if we use polar coordinates, the boundary is super cute:

$$r = \rho, \quad \theta = 0, \quad \theta = \alpha.$$

So, it makes a lot more sense to use these coordinates. To proceed, we need to *write the operator using polar coordinates also!* I leave it as a fun exercise involving the chain rule to prove that

$$\Delta = -\partial_{rr} - r^{-1}\partial_r - r^{-2}\partial_{\theta\theta}.$$

Let us try to solve $\Delta S = \lambda S$ in the circular sector using separation of variables. So, we have

$$R(r) \text{ and } \Theta(\theta).$$

The first one only depends on the r coordinate, whereas the second one only depends on the θ coordinate. Now, our PDE is:

$$-R''(r)\Theta(\theta) - r^{-1}R'(r)\Theta(\theta) - r^{-2}\Theta''(\theta)R(r) = \lambda R(r)\Theta(\theta).$$

First, we multiply everything by r^2, then we divide it all by ΘR to get

$$\frac{-r^2R'' - rR'}{R} - \frac{\Theta''}{\Theta} = \lambda \implies \frac{-r^2R'' - rR'}{R} - \lambda r^2 = \frac{\Theta''}{\Theta}.$$

Since the two sides depend on different variables, they are both constant. It turns out that the Θ side is much easier to deal with, so we look at solving it:

$$\frac{\Theta''}{\Theta} = \mu, \quad \Theta(0) = \Theta(\alpha) = 0.$$

Do you remember how to solve this? We have done it many times by now. I leave it as an *exercise* to show that there are no solutions (other than $\Theta = 0$, which is not allowed) for $\mu \ge 0$. Moreover, the (unnormalized) solutions for $\mu < 0$ are

$$\Theta_m(\theta) = \sin\left(\frac{m\pi\theta}{\alpha}\right), \quad \mu_m = -\frac{m^2\pi^2}{\alpha^2}.$$

As a consequence, we get the equation for R,

$$\frac{-r^2R''-rR'}{R}-\lambda r^2=\mu_m.$$

We multiply this equation by R, obtaining

$$-r^2R''-rR'-\lambda r^2R=\mu_mR.$$

This is equivalent to

$$r^2R''+rR'+(\lambda r^2+\mu_m)R=0.$$

We make a small clever change of variables. Let

$$x=\sqrt{\lambda}r,\quad f(x):=R(r),\quad r=\frac{x}{\sqrt{\lambda}}.$$

Then by the chain rule

$$R'(r)=\sqrt{\lambda}f'(x),\quad R''(r)=\lambda f''(x).$$

So, the equation becomes

$$\left(\frac{x^2}{\lambda}\right)\lambda f''(x)+\frac{x}{\sqrt{\lambda}}\sqrt{\lambda}f'(x)+(x^2+\mu_m)f(x)=0.$$

This simplifies, recalling that $\mu_m=-m^2\pi^2/\alpha^2$,

besseleq

$$x^2f''(x)+xf'(x)+(x^2-m^2\pi^2/\alpha^2)f(x)=0. \tag{1.1}$$

This is the definition of Bessel's equation of order $\frac{m\pi}{\alpha}$.

1.2. **Solving Bessel's equation.** In general, for notational convenience consider the equation

$$x^2f''+xf'+(x^2-\nu^2)f=0.$$

Assume that f has a series expansion (we will later see that this assumption luckily works out - if it didn't - we'd just have to keep trying other methods). Then we write

$$f(x)=\sum_{j\geq 0}a_jx^{j+b}.$$

Stick it into the ODE:

$$x^2\sum_{j\geq 0}a_j(j+b)(j+b-1)x^{j+b-2}+x\sum_{j\geq 0}a_j(j+b)x^{j+b-1}+(x^2-\nu^2)\sum_{j\geq 0}a_jx^{j+b}=0.$$

Pull the factors of x inside the sum:

$$\sum_{j\geq 0}a_j(j+b)(j+b-1)x^{j+b}+\sum_{j\geq 0}a_j(j+b)x^{j+b}+\sum_{j\geq 0}a_jx^{j+b+2}-\nu^2a_jx^{j+b}=0.$$

Begin with $j=0$. To make the sum vanish, it will certainly suffice to make all the individual terms in the sum vanish. So we would like to have

$$a_0\left(b(b-1)+b-\nu^2\right)x^b=0.$$

This will be true if

$$a_0=0 \text{ or } b^2-\nu^2=0 \implies b=\pm\nu.$$

Next look at $j=1$. We need

$$a_1\left((1+b)(1+b-1)+(1+b)-\nu^2\right)x^{b+1}=0.$$

Let's simplify what's in the parentheses, so we need

$$a_1\left((1+b)^2-\nu^2\right)=0.$$

So, here are our options:

(1) Let $b = \nu$, set $a_1 = 0$, and be free to choose a_0 OR
(2) Let $(1 + b) = \nu$, set $a_0 = 0$, and be free to choose a_1.

If we think about it, the second option is rather like doing the first one for $\nu - 1$ instead of ν. So, the two options are basically equivalent, but the first one is a bit more simple, so that is what we choose to do. We set $b = \nu$, $a_1 = 0$, and we shall choose $a_0 \neq 0$ later.

What happens with the higher terms? Once $j \geq 2$ the term with $a_j x^{j+b+2}$ gets involved. Let's group the terms in the series in a nice way:

$$\sum_{j \geq 0} x^{j+b} a_j \left((j+b)(j+b-1) + (j+b) - \nu^2 \right) + a_j x^{j+b+2} = 0.$$

This is

$$\sum_{j \geq 0} x^{j+b} a_j \left((j+b)^2 - \nu^2 \right) + a_j x^{j+b+2} = 0.$$

We figured out how to make the terms with the powers x^b and x^{b+1} vanish. For the higher powers, the coefficient of

$$x^{j+b+2} \text{ is } a_{j+2} \left((j+2+b)^2 - \nu^2 \right) + a_j.$$

Therefore, we need

$$a_{j+2} \left((j+2+b)^2 - \nu^2 \right) = -a_j \implies a_{j+2} = -\frac{a_j}{(j+2+b)^2 - \nu^2)}.$$

Recalling that we picked $b = \nu$, this means

$$a_{j+2} = -\frac{a_j}{(j+2+\nu)^2 - \nu^2},$$

so we are not dividing by zero which is a good thing. Equivalently, for $j \geq 2$, we have

$$a_j = -\frac{a_{j-2}}{(j+\nu)^2 - \nu^2} = -\frac{a_{j-2}}{j^2 + 2\nu j} = -\frac{a_{j-2}}{j(j+2\nu)}.$$

We therefore see that since we picked $a_1 = 0$, all of the odd terms are zero. On the other hand, for the even terms, we can figure out what these are using induction. I claim that

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (1+\nu)(2+\nu) \ldots (k+\nu)}.$$

To begin we check the base case which has $k = 1$:

$$a_2 = -\frac{a_0}{2(2+2\nu)} = -\frac{a_0}{4(1+\nu)} = \frac{(-1)^1 a_0}{2^{2(1)} 1! (1+\nu)}.$$

So the formula is correct. We next assume that it holds for k and verify using what we computed above that it works for $k + 1$. We have for $j = 2k + 2$,

$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+2+2\nu)}.$$

We insert the expression for a_{2k} by the induction assumption that the formula holds for k:

$$a_{2k+2} = -\frac{(-1)^k a_0}{(2k+2)(2k+2+2\nu) 2^{2k} k! (1+\nu)(2+\nu) \ldots (k+\nu)}.$$

We note that

$$(2k+2)(2k+2+2\nu) = 4(k+1)(k+1+\nu) = 2^2(k+1)(k+1+\nu).$$

So

$$a_{2k+2} = -\frac{(-1)^k a_0}{2^{2(k+1)}(k+1)k!(1+\nu)(2+\nu)\dots(k+\nu)(k+1+\nu)}.$$

Finally we note that

$$(k+1)k! = (k+1)!.$$

So,

$$a_{2k+2} = -\frac{(-1)^k a_0}{2^{2(k+1)}(k+1)!(1+\nu)(2+\nu)\dots(k+\nu)(k+1+\nu)}.$$

This is the formula for $k+1$, so it is indeed correct. Before we proceed, we recall one of the many special functions,

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \quad s \in \mathbb{C}, \quad \Re(s) > 1.$$

Using integration by parts, we compute

$$\Gamma(s) = \frac{t^s}{s}e^{-t}\Big|_0^\infty + \int_0^\infty \frac{t^s}{s}e^{-t}dt = \frac{1}{s}\Gamma(s+1).$$

Hence we see that

$$s\Gamma(s) = \Gamma(s+1).$$

Moreover, we compute directly that

$$\Gamma(1) = 1.$$

Therefore,

$$\Gamma(2) = 1, \quad \Gamma(3) = 2\Gamma(2) = 2, \quad \Gamma(n+1) = n!, \quad n \in \mathbb{Z}.$$

It is for this reason we define

$$0! := 1.$$

Moreover, viewing Γ as an extension of the factorial function to real numbers, we can compute silly expressions like

$$\pi! = \Gamma(\pi+1), \quad e! = \Gamma(e+1), \quad i! = \Gamma(i+1).$$

It gets better: we can extend Γ to be a meromorphic function on $\mathbb{C} \setminus -\mathbb{N}$. As long as $s \in \mathbb{C}$, $s \notin -\mathbb{N}$, we have

$$\Gamma(s) = \frac{1}{s}\Gamma(s+1).$$

So, we start with Γ defined for $\Re(s) > 1$, but then we can extend to $\Re(s) > 0$ by the above. Rinse and repeat. The only problem when we extend to $\Re(s) > 0$ is that $\Gamma(0) = \frac{1}{0}\Gamma(1) = \frac{1}{0}$ uh yeah. There is a simple pole at $s = 0$. When we repeat the extension procedure, the pole then appears again at $s = -1$. It continues along all the negative integers. However, except for these points, Γ is holomorphic. On the other hand, it is a bit non-trivial but possible to prove that $\frac{1}{\Gamma}$ is an *entire* function. I leave that as an exercise for those complex analysis fans ♡

So, motivated by the form of the coefficients, the tradition is to choose

$$a_0 = \frac{1}{2^\nu\Gamma(\nu+1)}.$$

Therefore coefficient

$$a_{2k} = \frac{(-1)^k}{2^{2k+\nu} k!(1+\nu)(2+\nu)\dots(k+\nu)\Gamma(\nu+1)} = \frac{(-1)^k}{2^{2k+\nu} k!\Gamma(k+\nu+1)}.$$

This is because

$$(\nu+1)\Gamma(\nu+1) = \Gamma(\nu+2).$$

Next

$$(\nu+2)\Gamma(\nu+2) = \Gamma(\nu+3).$$

We continue all the way to

$$(\nu+k)\Gamma(\nu+k) = \Gamma(\nu+k+1).$$

We have therefore arrived at the definition of the Bessel function of order ν,

$$J_\nu(x) := \sum_{k \geq 0} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

For the special case $\nu = n \in \mathbb{N}$, the Bessel function is defined for good reason via

$$J_{-n}(x) = (-1)^n J_n(x).$$

The Weber Bessel function is defined for $\nu \notin \mathbb{N}$ to be

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

The second linearly independent solution to Bessel's equation is then defined for $n \in \mathbb{N}$ to be

$$Y_n(x) := \lim_{\nu \to n} Y_\nu(x),$$

and this is well defined. If you are curious about Bessel functions, there are books by Olver, Watson, and Lebedev to name a few. What is most important about Y_n is that it blows up when $x \to 0$. That's okay. Since $J_n(x) \to 0$ as $x \to 0$, for $n \geq 1$, this shows that Y_n and J_n are certainly linearly independent! Hence they indeed form a basis of solutions to the Bessel equation.

Let us now return to our original problem. We can solve (1.1) with

$$f(x) = J_{m\pi/\alpha}(x) = J_{m\pi/\alpha}(\sqrt{\lambda} r).$$

Hence we have found a partner function to $\Theta_m(\theta)$, that is

$$R_m(r) = J_{m\pi/\alpha}(\sqrt{\lambda} r).$$

You are probably wondering WHAT IS LAMBDA? This comes from the boundary condition. We had the boundary condition that

$$R_m(0) = R_m(\rho) = 0.$$

So, for this reason we take the positive $m\pi/\alpha$ rather than the negative, because this guarantees that $R_m(0) = 0$, whereas taking $-m\pi/\alpha$, the Bessel function blows up at 0. Moreover, we also have the condition

$$R_m(\rho) = 0.$$

Well, that means we need

$$J_{m\pi/\alpha}(\rho\sqrt{\lambda}) = 0.$$

Therefore

$$\sqrt{\lambda}\rho \text{ needs to be a zero of } J_{m\pi/\alpha}.$$

If you look at the formula for the Bessel function, it looks a lot similar to a sine or cosine, because

$$\sin(x) = \sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos(x) = \sum_{k\geq 0} \frac{(-1)^k x^{2k}}{(2k)!}.$$

We know that these functions oscillate up and down, and the sine has zeros at the points $n\pi$ for $n \in \mathbb{Z}$, whereas the cosine has zeros at the points $(n+1/2)\pi$ for $n \in \mathbb{Z}$. So they go up and down and have these nice zero points dotting along the real line. One can think of the Bessel functions as the redneck cousins of the sine and cosine. Sine and cosine stayed in Sweden, but the Bessel functions took a boat to the USA about a hundred years ago. Although they have grown apart over these years, they still share many things in common with their Swedish sine and cosine relatives. For example, they love knäckebröd (especially with peanut butter) and they insist on everyone taking their shoes off inside the house! The Bessel functions share many general properties similar to those of the sine and cosine. For example, they have a discrete set of zeros which look similar to those of sine and cosine. Let

$$z_k(\nu) := \text{ the } k^{th} \text{ zero of the Bessel function of order } \nu.$$

Then for $\nu > 0$, we have

$$z_0(\nu) = 0 < z_1(\nu) < z_2(\nu) < \ldots \uparrow \infty.$$

So, to satisfy the boundary condition, we need

$$\lambda = \lambda_{k,m} = \frac{\left(z_k(m\pi/\alpha)\right)^2}{\rho^2}.$$

This ensures that

$$R_m(\rho) = J_{m\pi/\alpha}(\sqrt{\lambda_{k,m}}\rho) = J_{m\pi/\alpha}(z_k(m\pi/\alpha)) = 0,$$

because by definition, $z_k(m\pi/\alpha)$ is a zero of $J_{m\pi/\alpha}$.

Next time we will build up the full solution to the IVP for the heat equation and see how we can similarly solve the wave equation. We will also prove fun facts about Bessel functions!