FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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We continue into the world of Bessel functions with our motivating example of solving the heat (and wave) equations on a circular sector with the Dirichlet boundary condition.

1.1. Bessel functions for Dirichlet BC.

Bessel functions for Dirichlet BC. We shall not prove this theorem, but we may wish to use it at times.

Theorem 1. *The set of functions*

$$\Theta_m(\theta) J_{m\pi/\alpha}\left(\frac{z_{m,k} r}{\rho}\right), \quad k \geq 0, \quad m \geq 1$$

are an orthogonal basis for $\mathcal{L}^2$ on the sector of radius ρ and opening angle α. Above, $z_{m,k}$ is the k^{th} positive zero of $J_{m\pi/\alpha}$.

Using the theorem, we may now solve the IVP for both the heat equation as well as the wave equation on a sector when we have the Dirichlet boundary condition. We shall have earned ourselves a weekend once we do this. First we consider the IVP for the heat equation on the sector:

$$u_t + \Delta u = 0, \quad \text{inside the sector,}$$

$$u(r, \theta, 0) = v(r, \theta) \text{ inside the sector}$$

$$u = 0 \text{ on the boundary of the sector, and the same is true for } v.$$

The functions

$$S_{m,k}(\theta, r) := \Theta_m(\theta) J_{m\pi/\alpha}\left(\frac{z_{m,k} r}{\rho}\right)$$

satisfy

$$\Delta S_{m,k} = \lambda_{m,k} = \left(\frac{z_{m,k}}{\rho}\right)^2 S_{m,k},$$

because that's how we found them in the first place! Moreover, they are zero on the boundary of the sector because we constructed them to be so. The partner function

$$T_{m,k}(t)$$

satisfies

$$\frac{T'_{m,k}(t)}{T_{m,k}(t)} + \frac{\Delta S_{m,k}}{S_{m,k}} = 0 \implies T'_{m,k}(t) = -\lambda_{m,k} T_{m,k}(t).$$

So, up to constant factors

$$T_{m,k}(t) = e^{-\lambda_{m,k} t}.$$

As before, the constant factors are going to be the Fourier coefficients of our initial data. The solution is therefore

$$\sum_{m,k} e^{-\lambda_{m,k} t} \widehat{v_{m,k}} S_{m,k}(r, \theta).$$

Since we did not normalize the functions $S_{m,k}$ the coefficients are

$$\widehat{v_{m,k}} = \frac{\int_0^\rho \int_0^\alpha v(r,\theta) S_{m,k}(r,\theta) r dr d\theta}{\int_0^\rho \int_0^\alpha |S_{m,k}(r,\theta)|^2 r dr d\theta}.$$

Here we are using that amazing theorem above to know that we *can* expand v in terms of the functions $S_{m,k}$. In other word, the theorem guarantees that these functions actually comprise a fully complete basis for the Hilbert space $\mathcal{L}^2$ on the sector.

Finally we similarly solve the IVP for the wave equation on the sector with the additional initial condition that

$$u_t(r, \theta, 0) = 0.$$

The partner function now satisfies

$$\frac{T''_{m,k}(t)}{T_{m,k}(t)} + \frac{\Delta S_{m,k}}{S_{m,k}} = 0 \implies T''_{m,k}(t) = -\lambda_{m,k} T_{m,k}(t).$$

Therefore,

$$T_{m,k}(t) = a_{m,k} \cos(\sqrt{\lambda_{m,k}} t) + b_{m,k} \sin(\sqrt{\lambda_{m,k}} t).$$

To determine the coefficients, we use the ICs. The easiest one is demanding the time derivative to be zero, so we're going to need all of the sine terms gone. To help see this, just compute

$$T'_{m,k}(0) = \sqrt{\lambda_{m,k}} b_{m,k}.$$

We need this to be zero, but the λ part is not zero. So we need $b_{m,k} = 0$ for all m and k. So, our solution is

$$u(r, \theta, t) = \sum_{m,k} a_{m,k} \cos(\sqrt{\lambda_{m,k}} t) S_{m,k}(r, \theta),$$

with (from the IC)

$$a_{m,k} = \widehat{v_{m,k}} = \frac{\int_0^\rho \int_0^\alpha v(r,\theta) S_{m,k}(r,\theta) r dr d\theta}{\int_0^\rho \int_0^\alpha |S_{m,k}(r,\theta)|^2 r dr d\theta}.$$

1.2. **Bessel functions for Neumann boundary condition.** This theorem is another type of "adult spectral theorem."

Theorem 2. *Assume that $\nu \geq 0$ and $\rho > 0$. Assume that $c \geq -\nu$. Let*
$$\{z_k\}_{k\geq 1}$$
be the positive zeros of $cJ_\nu(x) + xJ'_\nu(x)$, and let
$$\psi_k(x) = J_\nu(z_k x/\rho).$$
If $c > -\nu$ then $\{\psi_k\}_{k\geq 1}$ is an orthogonal basis for $\mathcal{L}^2_w$ on the interval $(0, b)$ for the weight function $w(x) = x$. If $c > -\nu$, then $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w$ on the interval $(0, b)$ for the weight function $w(x) = x$, with $\psi_0(x) = x^\nu$.

Let us see how to apply this theorem when we are solving the heat (and wave) equations with the Neumann boundary condition. We follow the same procedure as for the heat equation. Let us name the sector
$$\Sigma.$$
$$u_t + \Delta u = 0, \quad \text{inside } \Sigma,$$
$$u(r,\theta,0) = v(r,\theta) \text{ inside } \Sigma$$
$$\text{the outward pointing normal derivative of } u = 0 \text{ on the boundary of } \Sigma.$$
We do the same procedure as before. We arrive at the equation for the Θ part:
$$\Theta'' = \mu\Theta, \quad \Theta'(0) = \Theta'(\alpha) = 0.$$
You can do the exercise to show that the only solutions are for $\mu < 0$, and to satisfy the boundary conditions, up to constant multiples
$$\Theta_m(\theta) = \sin(m\pi/\alpha), \quad \mu_m = -\frac{m^2\pi^2}{\alpha^2}, \quad m > 0.$$
Then, we again arrive at the Bessel equation of order $m\pi/\alpha$ for the function R. So, we get that
$$R_m(r) = J_{\nu_m}(\sqrt{\lambda}r), \quad \nu_m = m\pi/\alpha.$$
The boundary condition for R_m is that
$$R'_m(\rho) = 0.$$
So, this means we need
$$\sqrt{\lambda}J'_{\nu_m}(\sqrt{\lambda}\rho) = 0.$$
In other words, $\sqrt{\lambda}$ needs to be a solution of the equation
$$xJ'_{\nu_m}(\rho x) = 0.$$
If z_k is a solution to
$$xJ'_{\nu_m}(x) = 0,$$
then
$$z_k J'_{\nu_m}(z_k) = 0 \implies \frac{z_k}{\rho} J'_{\nu_m}(z_k\rho/\rho) = 0.$$
So, to satisfy the boundary condition, we need
$$\sqrt{\lambda} = \frac{z_k}{\rho} \implies \sqrt{\lambda}J'_{\nu_m}(\sqrt{\lambda}\rho) = 0.$$
Really, z_k also depends on m, so that is why we write $z_{m,k}$ to mean the k^{th} positive solution of the equation
$$xJ'_{\nu_m}(x) = 0.$$

Our function

$$R_{m,k}(r) = J_{\nu_m}(z_{m,k}r/\rho).$$

This also shows that

$$\lambda_{m,k} = \frac{z_{m,k}^2}{\rho^2}.$$

Now, we recall the equation for the partner function, T,

$$T'_{m,k}(t) = -\lambda_{m,k}T_{m,k}(t).$$

So, up to constant factors,

$$T_{m,k}(t) = e^{-\lambda_{m,k}t}.$$

To apply the theorem, we note that

$$\nu_m = m\pi/\alpha > 0 \forall m \in \mathbb{N}.$$

Therefore taking $c = 0$ in the theorem, $c \geq -\nu_m$ for all m. The theorem then tells us that the set

$$\{R_{m,k}(r)\}_{k\geq 1} = \{J_{\nu_m}(z_{m,k}r/\rho)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}^2(0,\rho)$ with respect to integrating against rdr. We also know that the $\Theta_m(\theta)$ functions are an orthogonal basis for $\mathcal{L}^2(0,\alpha)$ with respect to integrating against $d\theta$. Consequently, the entire collection

$$S_{m,k}(r,\theta) = \Theta_m(\theta)R_{m,k}(r)$$

is an orthogonal basis for $\mathcal{L}^2(\Sigma)$. This is because integrating on $\mathcal{L}^2(\Sigma)$ in polar coordinates is integrating

$$\int_\Sigma v(r,\theta)rdrd\theta = \int_0^\rho \int_0^\alpha v(r,\theta)rdrd\theta.$$

So, the theorem says that we can expand the initial data in a Fourier series with respect to the orthogonal basis functions $S_{m,k}$. We therefore write the solution

$$u(r,\theta,t) = \sum_{m,k} \widehat{v_{m,k}} T_{m,k}(t)S_{m,k}(r,\theta),$$

where

$$\widehat{v_{m,k}} = \frac{\int_\Sigma v(r,\theta)S_{m,k}(r)rdrd\theta}{||S_{m,k}||^2}$$

$$= \frac{\int_0^r \int_0^\theta \sin(m\pi\theta/\alpha)J_{m\pi/\alpha}(z_{m,k}r/\rho)v(r,\theta)rdrd\theta}{\int_0^r \int_0^\theta \sin(m\pi\theta/\alpha)^2 J_{m\pi/\alpha}(z_{m,k}r/\rho)^2 rdrd\theta}.$$

1.3. **Heat and waves in a disk.** Here is yet another adult spectral theorem.

Theorem 3. *Let $z_{n,k}$ be the positive zeros of $J_{|n|}(x)$. Then*

$$\{J_{|n|}(z_{n,k}r/\rho)e^{in\theta}\}_{n\geq 0, k\geq 1}$$

is an orthogonal basis for the disk of radius ρ. Moreover, let $w_{n,k}$ be the non-negative solutions to

$$xJ'_{|n|}(x) = 0.$$

Then,

$$\{J_{|n|}(w_{n,k}r/\rho)e^{in\theta}\}_{n,k\geq 0}$$

is an orthogonal basis for the disk of radius ρ.

Folland claims to give a proof, but it is bogus. The reason it's bogus is because it implicitly relies on Theorem 5.3 which is not proven (he refers to Watson). So rather than a bogus proof, the term adult spectral theorem reminded me of Adult Swim, a late night thing on comedy central, and that reminded me of Space Ghost, and this is one of the funniest episodes `https://www.youtube.com/watch?v=tB1ZjriDPd0` It also happens to feature an awesome nordic musician.

So, back to business. Let us use in this example the wave equation on the disk of radius ρ. Assume we are solving the homogeneous PDE with some initial condition. The first case corresponds to the Dirichlet boundary condition, whereas the second case corresponds to the Neumann boundary condition. We are solving

$$u_{tt}(r,\theta,t) + \Delta u = 0, \quad u_t(r,\theta,0) = 0, \quad u(r,\theta,0) = v(r,\theta).$$

1.3.1. *DBC.* First consider the Dirichlet boundary condition:

$$u(\rho,\theta,t) = 0.$$

This is because the disk only has boundary at the circular edge (no straight edges like sectors have). However, because we're on a disk, the function must be 2π periodic in the θ variable:

$$u(\rho,\theta+2\pi,t) = u(\rho,\theta,t).$$

When we separate variables, we end up with the equation for Θ:

$$\Theta'' = \mu\Theta, \quad \Theta(\theta) = \Theta(\theta+2\pi).$$

You may repeat the calculations which show that the only solutions are for $\mu < 0$ with $\mu \in \mathbb{Z}$ and

$$\Theta_n(\theta) = e^{in\theta}.$$

Next we proceed to the R part of the solution. The equation for R turns into the Bessel equation exactly as before, so we have

$$R_n(\sqrt{\lambda}r) = J_{|n|}(\sqrt{\lambda}r).$$

For the DBC we need

$$J_{|n|}(\sqrt{\lambda}\rho) = 0 \implies \sqrt{\lambda} = z_{n,k}/\rho.$$

Hence we have the function

$$R_{n,k}(r) = J_{|n|}(z_{n,k}r/\rho).$$

The partner functions in time are

$$T_{n,k}(t) = a_{n,k}\cos(z_{n,k}t/\rho) + b_{n,k}\sin(z_{n,k}t/\rho).$$

The theorem says that the functions

$$S_{n,k}(r,\theta) = \Theta_n(\theta)R_{n,k}(r) = e^{in\theta}J_{|n|}(z_{n,k}r/\rho)$$

are an orthogonal basis for the disk. So, we should be able to find coefficients $a_{n,k}$ and $b_{n,k}$ to solve the problem. Since we want u_t to vanish at $t = 0$, we take $b_{n,k} = 0$ for all n and k. Then, we have

$$u(r,\theta,t) = \sum_{n\in\mathbb{Z}} a_{n,k}\cos(z_{n,k}t/\rho)S_{n,k}(r,\theta).$$

To satisfy the IC, we take

$$a_{n,k} = \widehat{v_{n,k}} = \frac{\int_0^\rho \int_0^{2\pi} v(r,\theta)S_{n,k}(r,\theta)rdrd\theta}{||S_{n,k}||^2}.$$

By the magical theorem, the functions $S_{n,k}$ are an orthogonal basis, so indeed we can expand the initial data, v in terms of this basis.

1.3.2. *NBC.* For the Neumann boundary condition, everything proceeds in the same way up to the point where we specify $\sqrt{\lambda}$. In this case we need

$$\sqrt{\lambda} J'_{|n|}(\sqrt{\lambda}\rho) = 0.$$

If $w_{n,k}$ is a solution to the equation

$$xJ'_{|n|}(x) = 0,$$

then

$$\frac{w_{n,k}}{\rho} J'_{|n|}(w_{n,k}) = 0,$$

so

$$\sqrt{\lambda} = w_{n,k}/\rho \implies \sqrt{\lambda} J'_{|n|}(\sqrt{\lambda}\rho) = 0.$$

Hence our function

$$R_{n,k}(r) = J_{|n|}(w_{n,k} r/\rho).$$

The partner functions in time are

$$T_{n,k}(t) = a_{n,k} \cos(w_{n,k} t/\rho) + b_{n,k} \sin(w_{n,k} t/\rho).$$

We still have the IC which demands the time derivative vanishes at $t = 0$ hence all the sine terms are gone. To satisfy the other IC, we take

$$a_{n,k} = \widehat{v_{n,k}} = \frac{\int_0^\rho \int_0^{2\pi} v(r,\theta) S_{n,k}(r,\theta) r dr d\theta}{||S_{n,k}||^2}.$$

By the magical theorem, the functions $S_{n,k}$ are an orthogonal basis, so indeed we can expand the initial data, v in terms of this basis.

Having solved these problems on a disk, we can use separation of variables together with our current set of tools to solve the wave and heat equations on cylinders in three dimensions!!!

1.4. **Fun facts about Bessel functions.** To wrap up this Bessel function business, we prove some fun facts about them. This is just the tip of the iceberg when it comes to facts about Bessel functions.

Theorem 4 (Recurrence Formulas)**.** *For all x and ν*

$$(x^{-\nu} J_\nu(x))' = -x^{-\nu} J_{\nu+1}(x)$$

$$(x^{\nu} J_\nu(x))' = x^{\nu} J_{\nu-1}(x)$$

$$xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x)$$

$$xJ'_\nu(x) + \nu J_\nu(x) = xJ_{\nu-1}(x)$$

$$xJ_{\nu-1}(x) + xJ_{\nu+1}(x) = 2\nu J_\nu(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$$

Proof: Can you guess what we do? That's right - use the definition!!!! First,

$$x^{-\nu} J_\nu(x) = \sum_{n \geq 0} \frac{(-1)^n \frac{x^2 n}{2^{2n+\nu}}}{n! \Gamma(n+\nu+1)}.$$

Take the derivative of the sum termwise. This is totally legitimate because this series converges locally uniformly in $\mathbb{C}$. So, we compute

$$\sum_{n \geq 1} \frac{(-1)^n 2n \frac{x^{2n-1}}{2^{2n+\nu}}}{n! \Gamma(n+\nu+1)} = \sum_{m \geq 0} \frac{(-1)^{m+1} 2(m+1) \frac{x^{2m+1}}{2^{2m+2+\nu}}}{(m+1)! \Gamma(m+2+\nu)}.$$

Above we re-indexed the sum by defining $n = m+1$. Next we do some simplifying around

$$= -\sum_{m \geq 0} \frac{(-1)^m \frac{x^{2m+1}}{2^{2m+1+\nu}}}{m! \Gamma(m+2+\nu)} = -x^{-\nu} \sum_{m \geq 0} \frac{(-1)^m \frac{x^{2m+1+\nu}}{2^{2m+1+\nu}}}{m! \Gamma(m+2+\nu)} = -x^{-\nu} J_{\nu+1}(x).$$

Next we compute similarly the derivative of $x^\nu J_\nu$ is

$$\sum_{n \geq 0} \frac{(-1)^n (2n+2\nu) \frac{x^{2n+2\nu-1}}{2^{2n+\nu}}}{n! \Gamma(n+\nu+1)}.$$

We factor out a 2 to get

$$\sum_{n \geq 0} \frac{(-1)^n (n+\nu) \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}}{n! \Gamma(n+\nu+1)}.$$

Note that

$$\Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu) \implies \frac{(n+\nu)}{\Gamma(n+\nu+1)} = \frac{1}{\Gamma(n+\nu)}.$$

So, above we have

$$\sum_{n \geq 0} \frac{(-1)^n \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}}{n! \Gamma(n+\nu)} = x^\nu J_{\nu-1}(x).$$

To do the third one it is basically expanding out the first one:

$$(x^{-\nu} J_\nu(x))' = -\nu x^{-\nu-1} J_\nu + x^{-\nu} J_\nu' = -x^{-\nu} J_{\nu+1}.$$

Multiply through by $x^{\nu+1}$ to get

$$-\nu J_\nu + x J_\nu' = -x J_{\nu+1}.$$

We do similarly in the second formula:

$$\nu x^{\nu-1} J_\nu + x^\nu J_\nu' = x^\nu J_{\nu-1}.$$

Multiply by $x^{-\nu+1}$ to get

$$\nu J_\nu + x J_\nu' = x J_{\nu-1}.$$

Next, to get the fifth formula, subtract the third formula from the fourth. Finally, to get the sixth formula, add the third formula to the fourth.

We shall prove two lovely facts about the Bessel functions. The following fact is a theory item!

1.5. **The generating function for the Bessel functions.** This is a lovely, follow your nose and use the definitions type of proof.

Theorem 5. *For all x and for all $z \neq 0$, the Bessel functions, J_n satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}.$$

Proof. We begin by writing out the familiar Taylor series expansion for the exponential functions

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

These converge beautifully, absolutely and uniformly for z in compact subsets of $\mathbb{C} \setminus \{0\}$. So, since we presume that $z \neq 0$, we can multiply these series and fool around with them to try to make the Bessel functions pop out... Thus, we write

bessel1

$$e^{xz/2} e^{-x/(2z)} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k \geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}. \tag{1.1}$$

Here is where the one and only clever idea enters into this proof, but it's rather straightforward to come up with it. We would like a sum with $n = -\infty$ to ∞. So we look around into the above expression on the right, hunting for something which ranges from $-\infty$ to ∞. The only part which does this is $j - k$, because each of j and k range over 0 to ∞. Thus, we keep k as it is, and we let $n = j - k$. Then $j + k = n + 2k$, and $j = n + k$. However, now, we have $j! = (n + k)!$, but this is problematic if $n + k < 0$. There were no negative factorials in our original expression! So, to remedy this, we use the equivalent definition via the Gamma function,

$$j! = \Gamma(j + 1), \quad k! = \Gamma(k + 1).$$

Moreover, we observe that in (1.1), $j!$ and $k!$ are for j and k non-negative. We also observe that

$$\frac{1}{\Gamma(m)} = 0, \quad m \in \mathbb{Z}, \quad m \leq 0.$$

Hence, we can write

$$e^{xz/2} e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n + k + 1) k!}.$$

This is because for all the terms with $n + k + 1 \leq 0$, which would correspond to $(n+k)!$ with $n+k < 0$, those terms ought not to be there, but indeed, the $\frac{1}{\Gamma(n+k+1)}$ causes those terms to vanish!

Now, by definition,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k! \Gamma(k + n + 1)}.$$

Hence, we have indeed see that

$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} J_n(x)z^n.$$

$\square$

As an application of the theorem, we will determine an integral representation of the Bessel functions of integer order. We shall do this in the next lecture...