FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Let's look at another example. Consider a circular shaped rod. We can then use the coordinate $x \in [0, 2\pi]$ with 0 and 2π identified, for the position on the rod. We use the variable $t \geq 0$ for time. The function $u(x,t)$ is the temperature on the rod at position x at time t. The heat equation (with no sources or sinks) tells us that:

$$u_t = k u_{xx},$$

for some constant $k > 0$. By the same little time-units-trick, we can assume that $k = 1$. So, we use the "mathematician's heat equation,"

$$u_t = u_{xx}.$$

Let's see what happens when we try Technique 0, Separation of Variables. We write

$$u(x,t) = f(x)g(t).$$

Plug it into the heat equation:

$$g'(t)f(x) = f''(x)g(t).$$

We want to *separate variables,* so we want all the t-dependent bits on the left say, and all the x-dependent bits on the right. This can be achieved by dividing both sides by $f(x)g(t)$,

$$\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)}.$$

We now know that both sides must be constant (why?). So, which side do we choose first? Well, the g side has only one derivative, whereas the f side has a double derivative. One derivative (i.e. first order differential equation) is easier than double derivative (i.e. second order differential equation). So, we want to solve

$$\frac{g'(t)}{g(t)} = \lambda.$$

Equivalently, we can think about the left side as

$$(\log(g(t)))' = \lambda.$$

Integrating both sides, we get

$$\log g(t) = \lambda t + c.$$

Exponentiating both sides, we get

$$g(t) = e^c e^{\lambda t}.$$

Now, I've got a question for you physicists and chemists. You down-to-earth types. If there are no sources or sinks, what happens to heat over time? It disperses out into the universe right? So, can you tell me what is the only physically viable option for the constant, λ? That's right, we must have $\lambda \le 0$. Then, the equation for f is

$$\frac{f''(x)}{f(x)} = \lambda \le 0 \implies f''(x) = \lambda f(x).$$

In case $\lambda = 0$, $f(x) = ax + b$ is a linear function. Now, let's remember *where* x is. It's on a circle. So, we need $f(x) = f(x + 2\pi)$. That is only possible for a linear function if it's a constant function. Okay fine. We also consider $\lambda < 0$. Then our old multivariable calculus theorem tells us that a basis of solutions is

$$\{\sin(\sqrt{|\lambda|}x), \cos(\sqrt{|\lambda|}x)\}.$$

In order to make sure that $f(x) = f(x + 2\pi)$, we need $\sqrt{|\lambda|} \in \mathbb{N}$. Hence, we have found the solutions

$$g_n(t) = e^{-n^2 t}, \quad f_n(x) = a_n \cos(nx) + b_n \sin(nx), \quad u_n(x,t) = f_n(x) g_n(t), \quad n \in \mathbb{N}.$$

Since our equation

$$\partial_t u_n - \partial_{xx} u_n = 0$$

is satisfied for all n, the same is true for the sum,

$$\partial_t \sum_{n \ge 0} u_n(x,t) - \partial_{xx} \sum_{n \ge 0} u_n(x,t) = 0.$$

At this point, we're not sure about:

Question 1. *Given the initial temperatures on the rod, $v(x)$, at time $t = 0$, can we choose the constants a_n and b_n so that for*

$$u(x,t) := \sum_{n \ge 0} u_n(x,t),$$

we have

$$u(x,0) = v(x)?$$

Basically, we've found all the u_n's we could using the separation of variables technique. We are hoping that we can build the full solution out of these guys. Fourier made the bold guess that yep, we can build the full solution out of these guys. It took a long time to rigorously prove him right (like 100 years, because this whole theory about Hilbert spaces, measure theory, and functional analysis needed to get developed by Hilbert & his contemporaries).

1.1. **Introduction to Fourier Series of periodic functions.** For starters, we're going to consider periodic functions. Physically, this corresponds to heat and waves happening on anything which is circular (or even elliptical, wonky/warped circular) shaped. However, we'll see later that this whole assumption of a function being periodic can be lifted! It works for non-periodic functions defined on intervals just as well. So, don't be concerned that we're limiting ourselves by this periodicity assumption. We're just trying not to get overwhelmed, learning to walk before we try to run.

Definition 2. A function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p iff for all $x \in \mathbb{R}$, $f(x+p) = f(x)$.

For example, $\sin(x)$ is periodic with period 2π. Our heat equation examples, $f_n(x) = a_n \cos(nx) + b_n \sin(nx)$ are periodic with period $2\pi/n$. A small observation: I did not say the *minimal period is* p. For example, $\sin(x)$ also satisfies $\sin(x+4\pi) = \sin(x)$ for all $x \in \mathbb{R}$. So, $\sin(x)$ is also 4π periodic. In general, if a function is periodic with period p, then it's also periodic with period $2p$, $3p$, $\ldots np$ for any $n \in \mathbb{N}$ with $n \geq 1$.

Exercise 1. *Prove this (hint: induction!).*

We shall prove a super useful little lemma about periodic functions and their integrals.

Lemma 3 (Integration of periodic functions lemma)**.** *If f is periodic with period p then for any $a \in \mathbb{R}$*

$$\int_a^{a+p} f(x)dx$$

is the same.

Proof: If we think about it, we want to show that the function

$$g(a) := \int_a^{a+p} f(x)dx$$

is a constant function. This looks awfully similar to the fundamental theorem of calculus. In any decent proof, we need to use the hypotheses of the lemma. So, we're going to need to use the assumption that f is periodic with period p, which tells us that

$$f(a+p) - f(a) = 0.$$

Now, since we want to consider a as a variable, we don't want it at both the top and the bottom of the integral defining g. Instead, we can use linearity of integration to write

$$g(a) = \int_0^{a+p} f(x)dx - \int_0^a f(x)dx.$$

Then, using the fundamental theorem on each of the two terms on the right,

$$g'(a) = f(a+p) - f(a) = 0.$$

Above, we use the fact that f is periodic with period p. Hence, $g'(a) \equiv 0$ for all $a \in \mathbb{R}$. This tells us that g is a constant function, so its value is the same for all $a \in \mathbb{R}$.

So you survived a bit of theory, now let's return to our physical motivation! We wanted to find coefficients so that the $u(x,t)$ we found to solve the heat equation would match up with the initial data, $v(x)$. If it does, then (using some advanced PDE theory beyond the scope of this humble course), $u(x,t)$ is indeed THE solution to the heat equation with initial data $v(x)$. Hence, $u(x,t)$ actually tells us the temperature on the rod at position x at time t. Cool. So, setting $t=0$ in the definition of $u(x,t)$ we want

vx
$$v(x) = \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx). \tag{1.1}$$

It is totally equivalent to work with complex exponentials, because

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}.$$

Exercise 2. *Show that we can write $v(x)$ as a series above in (vx 1.1) if and only if we can write*

$$v(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Moreover, show that

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1, \quad c_n = \frac{1}{2}(a_n + ib_n), n \leq -1.$$

Finally, use this to show that

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n},\, n \geq 0, \quad b_n = i(c_n - c_{-n}),\, n \geq 0.$$

Okay, so we want to write $v(x)$ as a linear combination of the functions e^{inx}. We write vectors as linear combinations of basis vectors in linear algebra all the time. In fact, a function like $v(x)$ is basically just an infinite dimensional vector. So, you've graduated to "linear algebra for adults," in which your vectors are now infinite dimensional. [1] To continue with the linear algebra concept, we need a notion of scalar (or inner, same thing) product and hence also a notion of orthogonality. It turns out that the notion we need is

Definition 4. For two functions, f and g, which are real or complex valued functions defined on $[a,b] \subset \mathbb{R}$, we define their scalar product to be

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

We say that f and g are orthogonal if $\langle f, g \rangle = 0$. We define the $L^2([a,b])$ *norm* of a function to be

$$||f||_{L^2([a,b])} = \sqrt{\langle f, f \rangle}.$$

OBS! Learn this definition right now!!!! It is really important. Every detail:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx, \quad ||f||^2 = \langle f, f \rangle.$$

Now, if you wonder *why* it is defined this way, that is because defining things this way has the very pleasant consequence that it *works.* Meaning, when we define

[1] Grigori Rozenblioum, who taught this class for many years, and is in general an awesome mathematician, used to say "If you can pass this course, then you've earned the right to buy Vodka at Systembolaget, regardless of your actual age."

things this way, we are able to use the separation of variables technique to solve the PDEs which we want to solve.

Proposition 5. *On the interval* $[-\pi, \pi]$*, the functions*

$$f_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$$

are an orthonormal set with respect to the scalar product above.

Proof: First, we show that these guys are orthogonal. To do that, we just take $m \neq n$ and compute

$$\int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}dx.$$

Of course, the 2π factors don't matter. They're not going to make the inner product vanish! We recall of course that

$$\overline{e^{imx}} = e^{-imx}.$$

So, we compute,

$$\int_{-\pi}^{\pi} e^{ix(n-m)}dx.$$

Now, I claim that the function $e^{ix(n-m)}$ is 2π-periodic. We compute

$$e^{i(x+2\pi)(n-m)} = e^{ix(n-m)}e^{2\pi i(n-m)}.$$

If $n > m$, then $n - m \geq 1$, and so $e^{2\pi i(n-m)} = 1$, because $2\pi(n-m)$ is a positive integer multiple of 2π. So, it's still the same point on the unit circle in $\mathbb{C}$ as the point $1 = 1 + 0i$. On the other hand, if $n < m$, then

$$e^{2\pi i(n-m)} = \overline{e^{2\pi i(m-n)}} = \overline{1} = 1.$$

That's just using the same reasoning to say that $e^{2\pi i(m-n)} = 1$, and the complex conjugate of something real is itself. So, indeed it's periodic with period 2π. Then, when we compute the integral, we get

$$\left.\frac{e^{ix(n-m)}}{n-m}\right|_{x=-\pi}^{\pi}.$$

Note that $\pi = -\pi + 2\pi$. So,

$$e^{i\pi(n-m)} = e^{-i\pi(n-m)} \implies \left.\frac{e^{ix(n-m)}}{n-m}\right|_{x=-\pi}^{\pi} = 0.$$

We've proven now that these guys are orthogonal. Next we prove that the $L^2([-\pi, \pi])$ norm of these functions is equal to one. So, we compute

$$\int_{-\pi}^{\pi} \frac{e^{inx}}{\sqrt{2\pi}}\frac{\overline{e^{inx}}}{\sqrt{2\pi}}dx = \frac{2\pi}{2\pi} = 1.$$

So, they're an orthonormal set. We want them to actually be an orthonormal *basis,* so that we can write for any $v(x)$,

$$v(x) = \sum_{n\in\mathbb{Z}} c_n\varphi_n(x), \quad \varphi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}.$$

In analogue to linear algebra, we should expect the coefficients to be the scalar product of our function $v(x)$ with the basis functions (vectors), $\varphi_n(x)$. It is a little tedious to carry around the $\sqrt{2\pi}$ factors, so for this reason, we can also use $\{e^{inx}\}$, which is still an orthogonal set, it's just not orthonormal.

Definition 6. Assume f is periodic on $[-\pi, \pi]$ with period 2π. Define

$$c_n := \frac{1}{2\pi}\langle f, e^{inx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

These are the Fourier coefficients of f. The Fourier series of f is

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

So, the real question is, *when does the Fourier series actually converge to equal $f(x)$?* To get you warmed up to computing Fourier coefficients, try the following

Exercise 3. *If f is as in the definition and is also even, prove that $b_n = 0$ for all n. If f is as in the definition and is also odd, prove that $a_n = 0$ for all n. (Hint: If you forgot what a_n and b_n are, look at the previous exercise!).*

1.1.1. *Examples.* Consider the function $f(x) = |x|$. It satisfies $f(-\pi) = f(\pi)$. We can just *make* it 2π-periodic by extending it to $\mathbb{R}$ to satisfy $f(x + 2\pi) = f(x)$ for all x. The graph then looks like a zig-zag or sawtooth. We compute the Fourier coefficients:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|e^{-inx}dx, \quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|dx = \frac{2\pi^2}{2(2\pi)} = \frac{\pi}{2}.$$

So, we compute

$$\int_{-\pi}^{0} -xe^{-inx}dx, \quad \int_{0}^{\pi} xe^{-inx}dx.$$

We do substitution in the first integral to change it to

$$\int_0^{\pi} xe^{inx}dx = \left.\frac{xe^{inx}}{in}\right|_0^{\pi} - \int_0^{\pi}\frac{e^{inx}}{in}dx$$
$$= \frac{\pi e^{in\pi}}{in} - \frac{e^{in\pi}}{(in)^2} + \frac{1}{(in)^2}.$$

Similarly we also use integration by parts to compute

$$\int_0^{\pi} xe^{-inx}dx = \left.\frac{xe^{-inx}}{-in}\right|_0^{\pi} - \int_0^{\pi}\frac{e^{-inx}}{(-in)}dx$$
$$= \frac{\pi e^{-in\pi}}{-in} - \frac{e^{-in\pi}}{(-in)^2} + \frac{1}{(-in)^2}.$$

Adding them up and use the 2π periodicity, we get

$$\frac{2e^{in\pi}}{n^2} - \frac{2}{n^2} = \frac{2(-1)^n - 2}{n^2}.$$

Hence $c_n = 0$ if n is even, whereas $c_n = -\frac{4}{n^2}$ if n is odd. The Fourier series is therefore

$$\frac{\pi}{2} + \sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{4}{n^2}\right).$$

Exercise 4. *Use these calculations to compute the Fourier cosine series, that is the series*

$$\sum_{n\geq 0} a_n \cos(nx).$$

Next, consider the function $f(x) = x$ initially on the interval $]-\pi,\pi[$. We can extend it in a similar way to be 2π periodic, but it will then be discontinuous with jump discontinuities at odd-integer multiples of π.

Exercise 5. *Compute in the same way the Fourier coefficients of this function, that is, compute*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx \quad n \in \mathbb{Z}.$$

Use that calculation to show that $a_n = 0$ for all n, and then compute the Fourier sine series,

$$\sum_{n\geq 1} b_n \sin(nx).$$

Look at these two examples. Do the series converge? Do they converge absolutely? Compare and contrast them!