FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Theorem 1. *The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$, and*

$$||P_n||^2 = \frac{2}{2n+1}.$$

Proof: We first prove the orthogonality. Assume that $n > m$. Then, since they have this constant stuff out front, we compute

$$2^n n! 2^m m! \langle P_n, P_m \rangle = \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m dx.$$

Let us integrate by parts once:

$$= \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m \Big|_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

Consider the boundary term:

$$\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}}(x-1)^n(x+1)^n.$$

This vanishes at $x = \pm 1$, because the polynomial vanishes to order n whereas we only differentiate $n-1$ times. So, we have shown that

$$2^n n! 2^m m! \langle P_n, P_m \rangle = -\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

We repeat this $n-1$ more times. We note that for all $j < n$,

$$\frac{d^j}{dx^j}(x^2-1)^n \text{ vanishes at } x = \pm 1.$$

For this reason, all of the boundary terms from integrating by parts vanish. So, we just get

$$(-1)^n \int_{-1}^{1} (x^2-1) \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^m dx = (-1)^n \int_{-1}^{1} (x^2-1) \frac{d^n}{dx^n} \frac{d^m}{dx^m}(x^2-1)^m dx$$

Remember that $n > m$. We computed that $\frac{d^m}{dx^m}(x^2-1)^m$ is a polynomial of degree m. So, if we differentiate it more than m times we get zero. So, we're integrating zero! Hence it is zero.

For the second part, we use the formula we computed for

$$P_n(x) = \frac{1}{2^n n!} \sum_{k \geq n/2}^{n} (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

Differentiating n times gives us just the term with the highest power of x, so we have

$$\frac{d^n}{dx^n} P_n(x) = \frac{1}{2^n n!} n! \prod_{j=0}^{n-1} (2n-j) = \frac{(2n)!}{2^n n!}.$$

Consequently,

$$\begin{aligned}
\langle P_n, P_n \rangle = (-1)^n \frac{1}{2^n n!} \frac{(2n)!}{2^n n!} \int_{-1}^{1} (x^2-1)^n dx &= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx \\
&= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^{2k} dx \\
&= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^{n-k} \frac{x^{2k+1}}{2k+1} \binom{n}{k} \Bigg|_0^1 \\
&= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{2k+1} \\
&= \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{1}{2k+1}.
\end{aligned}$$

This looks super complicated. Apparently by some miracle of life

$$\int_0^1 (1-x^2)^n dx = \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+3/2)}.$$

Since

$$\langle P_n, P_n \rangle = (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx = \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (1-x^2)^n dx,$$

we get

$$\frac{\Gamma(n+1)\Gamma(1/2)2(2n)!}{2^{2n}(n!)^2\Gamma(n+3/2)}.$$

We use the properties of the Γ function together with the fact that $\Gamma(1/2) = \sqrt{\pi}$ to obtain

$$\frac{\sqrt{\pi}2(2n)!}{2^{2n} n!(n+1/2)\Gamma(n+1/2)}.$$

Let us consider

$$2(n+1/2)\Gamma(n+1/2) = (2n+1)\Gamma(n+1/2).$$

Next consider

$$2(n-1/2)\Gamma(n-1/2) = (2n-1)\Gamma(n-1/2).$$

Proceeding this way, the denominator becomes

$$2^n n!(2n+1)(2n-1)\ldots 1\sqrt{\pi}.$$

However, now looking at the first part

$$2^n n! = 2n(2n-2)(2n-4)\ldots 2.$$

So together we get
$$(2n+1)!\sqrt{\pi}.$$
Hence putting this in the denominator of the expression we had above, we have
$$\frac{\sqrt{\pi}2(2n)!}{(2n+1)!\sqrt{\pi}} = \frac{2}{2n+1}.$$

Corollary 2. *The Legendre polynomials are an orthogonal basis for $\mathcal{L}^2$ on the interval $[-1,1]$.*

Theorem 3. *The even degree Legendre polynomials $\{P_{2n}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$. The odd degree Legendre polynomials $\{P_{2n+1}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$.*

Proof: Let f be defined on $[0,1]$. We can extend f to $[-1,1]$ either evenly or oddly. First, assume we have extended f evenly. Then, since $f \in \mathcal{L}^2$ on $[0,1]$,
$$\int_{-1}^{1} |f_e(x)|^2 dx = 2\int_0^1 |f(x)|^2 dx < \infty.$$
Therefore f_e is in $\mathcal{L}^2$ on the interval $[-1,1]$. We have proven that the Legendre polynomials are an orthogonal basis. So, we can expand f_e in a Legendre polynomial series, as
$$\sum_{n\geq 0} \hat{f}_e(n) P_n,$$
where
$$\hat{f}_e(n) = \frac{\langle f_e, P_n\rangle}{||P_n||^2.}$$
By definition,
$$\langle f_e, P_n\rangle = \int_{-1}^{1} f_e(x)P_n(x)dx.$$
Since f_e is even, the product $f_e(x)P_n(x)$ is an *odd* function whenever n is odd. Hence all of the odd coefficients vanish. Moreover,
$$\langle f_e, P_{2n}\rangle = 2\int_0^1 f(x)P_{2n}(x))dx.$$
We also have
$$||P_{2n}||^2 = 2\int_0^1 |P_{2n}(x)|^2 dx.$$
Consequently
$$f = \sum_{n\in\mathbb{N}} \left(\frac{\int_0^1 f(x)P_{2n}(x)dx}{\int_0^1 |P_{2n}(x)|^2 dx} \right) P_{2n}.$$
We can also extend f oddly. This odd extension satisfies
$$\int_{-1}^{1} |f_o(x)|^2 dx = \int_{-1}^{0} |f_o(x)|^2 dx + \int_0^1 |f_o(x)|^2 dx = 2\int_0^1 |f_o(x)|^2 dx < \infty.$$

So, the odd extension is also in $\mathcal{L}^2$ on the interval $[-1, 1]$. We can expand f_o in a Legendre polynomial series, as

$$\sum_{n \geq 0} \hat{f}_o(n) P_n,$$

where

$$\hat{f}_o(n) = \frac{\langle f_o, P_n \rangle}{||P_n||^2}.$$

By definition,

$$\langle f_o, P_n \rangle = \int_{-1}^{1} f_o(x) P_n(x) dx.$$

Since f_o is odd, the product $f_o(x)P_n(x)$ is an *odd* function whenever n is *even*. Hence all of the even coefficients vanish. Moreover,

$$\langle f_o, P_{2n+1} \rangle = 2 \int_0^1 f(x) P_{2n+1}(x)) dx,$$

because the product of two odd functions is an even function. We also have

$$||P_{2n+1}||^2 = \int_{-1}^{0} |P_{2n+1}(x)|^2 dx + \int_0^1 |P_{2n+1}(x)|^2 dx = 2 \int_0^1 |P_{2n+1}(x)|^2 dx.$$

Consequently

$$f = \sum_{n \in \mathbb{N}} \left(\frac{\int_0^1 f(x) P_{2n+1}(x) dx}{\int_0^1 |P_{2n+1}(x)|^2 dx} \right) P_{2n+1}.$$

1.1. **Legendre polynomials origins story.** We consider spherical coordinates in $\mathbb{R}^3$. These coordinates are useful for solving PDEs inside spheres or pieces of spheres. The spherical coordinates are (r, θ, ϕ). The first coordinate, r tells us the distance of the point in $\mathbb{R}^3$ to the origin. The second coordinate, θ, tells us the angle of the point in the $x - y$ plane. The third coordinate, ϕ, tells the angle of the point in the z direction. So, if $\phi = 0$, the point is along the positive z-axis. If $\phi = \frac{\pi}{2}$, the point has z-coordinate equal to zero. If $\phi = \pi$, the point is along the negative z-axis. The standard coordinate are therefore

$$x = r \cos\theta \sin\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\phi.$$

To see how this work, draw some right triangles from different perspectives (will do in lecture!). By the chain rule, the Laplace operator

$$\Delta = -\partial_x^2 - \partial_y^2 - \partial_z^2 = -\partial_r^2 - \frac{2}{r}\partial_r - \frac{\sin\phi \partial_\phi^2 + \cos\phi \partial_\phi)}{r^2 \sin\phi} - \frac{\partial_\theta^2}{r^2 \sin^2\phi}.$$

Consider solving the Dirichlet problem inside a sphere. We would like $\Delta u = 0$. Since the natural coordinates on a sphere are the spherical coordinates, we write u as a product of three functions depending on the three spherical coordinates,

$$R(r)\Theta(\theta)\Phi(\phi).$$

Then, the PDE becomes

$$\Delta(R\Theta\Phi) = 0 \implies \frac{R''}{R} + \frac{2R'}{rR} + \frac{\Phi'' \sin\phi + \Phi' \cos\phi}{r^2 \sin\phi \Phi} + \frac{\Theta''}{r^2 \sin^2\phi \Theta} = 0.$$

Let us use φ for the variable, ϕ, and continue to use Φ for the function. We multiply by $r^2 \sin^2 \varphi$:

$$\frac{R'' r^2 \sin^2 \varphi}{R} + \frac{r \sin^2 \varphi 2R'}{R} + \frac{\sin \varphi (\Phi'' \sin \varphi + \Phi' \cos \varphi}{\Phi} + \frac{\Theta''}{\Theta} = 0.$$

Since it is the most simple, we move Θ to the other side:

$$\frac{R'' r^2 \sin^2 \varphi}{R} + \frac{r \sin^2 \varphi 2R'}{R} + \frac{\sin \varphi (\Phi'' \sin \varphi + \Phi' \cos \varphi}{\Phi} = -\frac{\Theta''}{\Theta}.$$

Therefore both sides are constant. We deal with Θ first. Conquer the weakest opponents first, so that they are not trying to attack from behind whilst one deals with the more significant threats. The equation for Θ is by far the simplest. For geometric reasons, Θ must be a 2π periodic function. Therefore

$$-\frac{\Theta''}{\Theta} = m^2, \quad m \in \mathbb{Z}, \quad \Theta_m(\theta) = e^{im\theta}.$$

We therefore can use this in the equation for the right side:

$$\frac{R'' r^2 \sin^2 \varphi}{R} + \frac{r \sin^2 \varphi 2R'}{R} + \frac{\sin \varphi (\Phi'' \sin \varphi + \Phi' \cos \varphi}{\Phi} = m^2.$$

We divide by $\sin^2 \varphi$ and move all the φ dependent terms to the right side, obtaining

$$\frac{R'' r^2 + 2rR'}{R} = \frac{m^2}{\sin^2 \varphi} - \left(\frac{\sin \varphi \Phi'' + \cos \varphi \Phi'}{\sin \varphi \Phi} \right).$$

Similarly, as both sides depend on different variables, both sides must be constant. So, we shall call the constant λ. We shall deal with the φ business first, doing a clever transformation. Let

$$s = \cos \varphi.$$

Then we note that $\cos : [0, \pi] \to [-1, 1]$ bijectively. We also have $\varphi = \arccos s$. Let

$$S(s) := S(\cos \varphi) = \Phi(\varphi).$$

Then by the chain rule,

$$\Phi'(\varphi) = -\sin \varphi S'(s), \quad \Phi''(\varphi) = -\cos \varphi S'(s) + \sin^2 \varphi S''(s).$$

By definition of s, and the fact that $\sin^2 + \cos^2 = 1$,

$$\Phi''(\varphi) = -sS'(s) + (1 - s^2) S''(s).$$

We therefore see that

$$\frac{\Phi''}{\Phi} = \frac{-sS' + (1 - s^2) S''}{\Phi}, \quad \frac{\Phi' \cos \varphi}{\Phi \sin \varphi} = \frac{-\sin \varphi \cos \varphi S'}{\sin \varphi S} = -\frac{sS'}{S}.$$

The equation for the φ variable side is then

$$\lambda = \frac{m^2}{1 - s^2} - \left(\frac{-sS' + (1 - s^2) S''}{S} - \frac{sS'}{S} \right) = \lambda.$$

We multiply by S and obtain

$$\frac{Sm^2}{1 - s^2} - \left(-2sS' + (1 - s^2) S'' \right) = \lambda S.$$

Observe that

$$-2sS' + (1 - s^2) S'' = \left[(1 - s^2) S' \right]'.$$

So, the equation is

legm
$$\frac{Sm^2}{1-s^2} - \left[(1-s^2)S'\right]' - \lambda S = 0. \tag{1.1}$$

If $m = 0$, this equation is

leg0
$$-\left[(1-s^2)S'\right]' - \lambda S = 0 \iff \left[(1-s^2)S'\right]' + \lambda S = 0. \tag{1.2}$$

Since $m \in \mathbb{Z}$, we would like to find solutions to this equation. The easiest case is the case when $m = 0$. It turns out that the Legendre polynomials solve this equation.

Theorem 4. *The Legendre polynomials solve*

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x) = 0.$$

In particular, they are eigenfunctions for the SLP $[(1-x^2)'u']' + \lambda u = 0$ *with eigenvalues* $\lambda = n(n+1)$.

We postpone the proof until the next lecture, so that we can keep focused on solving the Dirichlet problem on the sphere. For $m = 0$, the functions $P_n(s)$ solves the equation (1.2), with $\lambda_n = n(n+1)$. For the general case, I leave it as an exercise to verify that

$$P_n^m(s) := (1-s^2)^{|m|/2} \frac{d^{|m|}}{ds^{|m|}} P_n(s)$$

solves (1.2). Recalling that $s = \cos\varphi$, we have therefore found functions

$$\Theta_m(\theta) = e^{im\theta},$$

and

$P_n^m(\varphi) = (1-s^2)^{|m|/2} \dfrac{d^{|m|}}{ds^{|m|}} P_n(s)$ first compute the derivative, then set $s = \cos\varphi$.

Finally, we use the value of $\lambda = n(n+1)$ to solve for the function R:

$$\frac{R''r^2 + 2rR'}{R} = \lambda_n = n(n+1).$$

This becomes

$$R''r^2 + 2rR' - \lambda_n R = 0.$$

This is an Euler equation. We look for solutions of the form $R(r) = r^\alpha$. Putting such a function into the ODE,

$$\alpha(\alpha-1)r^\alpha + 2\alpha r^\alpha - \lambda_n r^\alpha = 0 \iff \alpha^2 + \alpha - \lambda_n = 0.$$

We solve the quadratic equation for

$$\alpha = \frac{-1 \pm \sqrt{1+4\lambda_n}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1+4n(n+1)}}{2}.$$

We do not want $R(r) \to \infty$ when $r \to 0$, so we choose the solution with the plus. We fiddle a little with this square root part:

$$\frac{\sqrt{1+4n(n+1)}}{2} = \sqrt{\frac{1}{4} + n(n+1)} = \sqrt{n^2 + n + \frac{1}{4}} = \sqrt{(n+1/2)^2} = n + 1/2.$$

Consequently

$$-\frac{1}{2} + \frac{\sqrt{1+4n(n+1)}}{2} = n.$$

We have therefore found

$$R_n(r) = r^n.$$

Up to constant factors, we have thus found the functions

$$u_{m,n}(r,\theta,\varphi) = r^n e^{im\theta} P_n^m(\cos\varphi),$$

which solve

$$\Delta u_{m,n} = 0$$

in the sphere. It just so happens that we can smash them all together and solve the Dirichlet problem in a sphere.

Theorem 5. *The solution to the Dirichlet problem in the unit sphere in $\mathbb{R}^3$, that is*

$$\Delta u = 0, \quad u(1,\theta,\varphi) = f(\theta,\varphi)$$

is

$$u(r,\theta,\varphi) = \sum_{n\geq 0, m\in\mathbb{Z}} \widehat{f_{n,m}} r^n e^{im\theta} P_n^m(\cos\varphi),$$

with

$$\widehat{f_{n,m}} = \frac{\int_0^\pi \int_0^{2\pi} f(\theta,\varphi) e^{-im\theta} P_n^m(\cos\varphi) d\theta \sin\varphi d\varphi}{2\pi ||P_n^m||^2} = \frac{\int_{-1}^1 \int_0^{2\pi} f(\theta,\arccos(s)) e^{-im\theta} P_n^m(s) d\theta ds}{2\pi ||P_n^m||^2}.$$

The functions

$$Y_{m,n}(\theta,\varphi) = e^{im\theta} P_n^m(\varphi)$$

are called *spherical harmonics.* One can show that

$$||P_n^m||^2 = \frac{(n+m)!2}{(n-m)!(2n+1)}, \quad n \geq |m|,$$

and that

$$||P_n^m||^2 = 0, \quad n < |m|.$$

We have deserved some comic relief. This shall be provided by the French song, Foux du Fa Fa, an exerpt from the series, Flight of the Conchords `https://www.youtube.com/watch?v=EuXdhow3uqQ`. Parlez-vous le français?