FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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We begin by proving the theorem which was previously postponed. The proof is a great exercise in using orthogonal polynomials.

Theorem 1. *The Legendre polynomials solve*

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x) = 0.$$

In particular, they are eigenfunctions for the SLP $[(1-x^2)'u']' + \lambda u = 0$ *with eigenvalues* $\lambda = n(n+1)$.

Proof: By the product rule,

$$[(1-x^2)P_n']' = -2xP_n' + (1-x^2)P_n''.$$

We compute the leading coefficient coming from

$$-2xP_n' - x^2P_n''.$$

We recall that

$$P_n(x) = \frac{1}{2^n n!}\sum_{k\geq n/2}^{n}(-1)^{n-k}\binom{n}{k}x^{2k-n}\prod_{j=0}^{n-1}(2k-j).$$

The highest order term comes from $k=n$, and it is

$$\frac{1}{2^n n!}x^n\prod_{j=0}^{n-1}(2n-j) = \frac{1}{2^n n!}x^n\frac{(2n)!}{n!}.$$

We therefore compute that

$$-2xP_n' - x^2P_n'' = -\frac{2n(2n)!x^n}{2^n(n!)^2} - \frac{n(n-1)(2n)!x^n}{2^n(n!)^2} = \frac{(2n)!x^n(-2n-n(n-1))}{2^n(n!)^2}$$

$$= -\frac{(2n)!x^n n(n+1)}{2^n(n!)^2}.$$

If we look back at the highest order term in P_n itself, this was

$$\frac{(2n)!x^n}{2^n(n!)^2}.$$

So we see that the highest order term in

$$[(1-x^2)P_n']' \text{ is } -n(n+1)\frac{(2n)!x^n}{2^n(n!)^2}.$$

Consequently

$$[(1-x^2)P_n']' + n(n-1)P_n \text{ is a polynomial of degree } n-1 \text{ or lower.}$$

We may therefore express this polynomial, call it q as a linear combination of the Legendre polynomials of degree up to $n-1$, that is

$$q = \sum_{j=0}^{n-1} c_j P_j.$$

Let us compute the coefficients:

$$c_j = \frac{\langle q, P_j \rangle}{||P_j||^2}.$$

We first compute using integration by parts and the vanishing of the boundary terms:

$$\int_{-1}^{1} [(1-x^2)P_n']' P_j dx = -\int_{-1}^{1} (1-x^2)P_n' P_j' dx = \int_{-1}^{1} [(1-x^2)P_j']' P_n dx.$$

Observe that $[(1-x^2)P_j']'$ is a polynomial of degree $j < n$. It can therefore be written as a linear combination of $P_0, \ldots, P_j$. Each of these are orthogonal to P_n. Hence this part vanishes. For the second part, we compute

$$\int_{-1}^{1} n(n+1)P_n(x)P_j(x)dx = 0,$$

since $j < n$. So in fact all together, $c_j = 0$ for all $j = 0, \ldots, n-1$. We therefore have computed that

$$[(1-x^2)P_n']' + n(n-1)P_n = 0.$$

1.1. **Hermite polynomials.**

Definition 2. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Proposition 3. *The Hermite polynomials are polynomials with the degree of H_n equal to n.*

Proof: The proof is by induction. For $n = 0$, this is certainly true, as $H_0 = 1$. Next, let us assume that

$$\frac{d^n}{dx^n} e^{-x^2} = p_n(x) e^{-x^2},$$

is true for a polynomial, p_n which is of degree n. Then,

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = \frac{d}{dx}\left(p_n(x)e^{-x^2}\right) = p_n'(x)e^{-x^2} - 2xp_n(x)e^{-x^2} = \left(p_n'(x) - 2xp_n(x)\right)e^{-x^2}.$$

Let

$$p_{n+1} = p_n'(x) - 2xp_n(x).$$

Then we see that since p_n is of degree n, p_{n+1} is of degree $n+1$. Moreover

$$\frac{d^{n+1}}{dx^{n+1}}e^{-x^2} = p_{n+1}(x)e^{-x^2}.$$

So, in fact, the Hermite polynomials satisfy:

$$H_0 = 1, \quad H_{n+1} = -\left(H_n'(x) - 2xH_n(x)\right).$$

Proposition 4. *The Hermite polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function e^{-x^2}. Moreover, with respect to this weight function $||H_n||^2 = 2^n n!\sqrt{\pi}$.*

Proof: Assume $n > m \geq 0$. We compute

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}dx = \int_{\mathbb{R}}(-1)^n \frac{d^n}{dx^n}e^{-x^2}H_m(x)dx.$$

We use integration by parts n times, noting that the rapid decay of e^{-x^2} kills all boundary terms. We therefore get

$$\int_{\mathbb{R}} e^{-x^2}\frac{d^n}{dx^n}H_m(x)dx = 0.$$

This is because the polyhomial, H_m, is of degree $m < n$. Therefore differentiating it n times results in zero. Finally, for $n = m$, we have by the same integration by parts,

$$\int_{\mathbb{R}} H_n^2(x)e^{-x^2}dx = \int_{\mathbb{R}} e^{-x^2}\frac{d^n}{dx^n}H_n(x)dx.$$

The n^{th} derivative of H_n is just the n^{th} derivative of the highest order term. By our preceding calculation, the highest order term in H_n is

$$(2x)^n.$$

Differentiating n times gives

$$2^n n!.$$

Thus

$$\int_{\mathbb{R}} H_n^2(x)e^{-x^2}dx = 2^n n!\int_{\mathbb{R}} e^{-x^2}dx = 2^n n!\sqrt{\pi}.$$

We may wish to use the following lovely fact, but we shall not prove it.

Theorem 5. *The Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-x^2}.*

What we shall prove, however, is a theory item concerning the Hermite polynomials.

1.1.1. *The generating function for the Hermite polynomials.* This is similar to the analogous result for the Bessel functions, but with a bit of a twist.

Theorem 6. *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

satisfy

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Proof: The key idea with which to begin is to consider instead

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2}.$$

We consider the Taylor series expansion of this guy, with respect to z, viewing x as a parameter. By definition, the Taylor series expansion for

$$e^{-(x-z)^2} = \sum_{n \geq 0} a_n z^n,$$

where

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} e^{-(x-z)^2}, \quad \text{evaluated at } z = 0.$$

To compute these coefficients, we use the chain rule, introducing a new variable $u = x - z$. Then,

$$\frac{d}{dz} e^{-(x-z)^2} = -\frac{d}{du} e^{-u^2},$$

and more generally, each time we differentiate, we get a -1 popping out, so

$$\frac{d^n}{dz^n} e^{-(x-z)^2} = (-1)^n \frac{d^n}{du^n} e^{-u^2},$$

Hence, evaluating with $z = 0$, we have

$$a_n = \frac{1}{n!} (-1)^n \frac{d^n}{du^n} e^{-u^2}, \quad \text{evaluated at } u = x.$$

The reason it's evaluated at $u = x$ is because in our original expression we're expanding in a Taylor series around $z = 0$ and $z = 0 \iff u = x$ since $u = x - z$. Now, of course, we have

$$\frac{d^n}{du^n} e^{-u^2}, \quad \text{evaluated at } u = x = \frac{d^n}{dx^n} e^{-x^2}.$$

Hence, we have the Taylor series expansion

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2} = \sum_{n \geq 0} \frac{z^n}{n!} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Now, we multiply both sides by e^{x^2} to obtain

$$e^{2xz-z^2} = e^{x^2} \sum_{n \geq 0} \frac{z^n}{n!} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

We can bring e^{x^2} inside because everything converges beautifully. Then, we have

$$e^{2xz-z^2} = \sum_{n \geq 0} \frac{z^n}{n!} e^{x^2} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Voilà! The definition of the Hermite polynomials is staring us straight in the face! Hence, we have computed

$$e^{2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!} H_n(x).$$

The Hermite polynomials come from solving PDEs in parabolic shaped regions of $\mathbb{R}^2$.

1.2. **The Laguerre polynomials.** The Laguerre polynomials come from understanding the quantum mechanics of the hydrogen atom. We shall not get into this[1]

Definition 7. The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!} \frac{d^n}{dx^n}(x^{\alpha+n}e^{-x}).$$

We summarize their properties in the following

Theorem 8 (Properties of Laguerre polynomials)**.** *The Laguerre polynomials are an orthogonal basis for $\mathcal{L}^2$ on $(0,\infty)$ with the weight function $x^\alpha e^{-x}$. Their norms squared,*

$$||L_n^\alpha||^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

They satisfy the Laguerre equation

$$[x^{\alpha+1}e^{-x}(L_n^\alpha)']' + nx^\alpha e^{-x} L_n^\alpha = 0.$$

For $x > 0$ and $|z| < 1$,

$$\sum_{n=0}^\infty L_n^\alpha(x) z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}}.$$

Now for what we've all been waiting for: applications to best approximations!

1.3. **Applications to best approximations.** Here is a typical problem: find the polynomial, $P(x)$, of at most degree 5 which minimizes

$$\int_a^b |f(x) - P(x)|^2 dx.$$

Here you would be explicitly given the function f as well as the interval from a to b. Since it works the same way, it seems wise to show the general principle. Then, you can use this for your particular problems. We know that the Legendre polynomials are an orthogonal basis for $\mathcal{L}^2$ on $(-1,1)$. Let's first assume

$$a = -1, \quad b = 1.$$

Then, we compute

$$c_n = \frac{\int_{-1}^1 f(x)P_n(x)dx}{||P_n||^2}, \quad n = 0,1,2,3,4,5.$$

[1] Alex Jones does get into it: `https://www.youtube.com/watch?v=i91XV07Vsc0`. Check out the Alex Jones Prison Planet `https://www.youtube.com/watch?v=kn_dHspHd8M`. Turns out that Alex Jones's crazy ranting makes for decent death metal vocals. The gay frogs and America first remix are pretty decent too.

The polynomial is, by the best approximation theorem(s),

$$P(x) = \sum_{n=0}^{5} c_n P_n(x).$$

So, suppose we don't have $a = -1$ and $b = 1$, but instead we've got some other interval. Let

$$m = \frac{a+b}{2}.$$

This is the midpoint of the interval. Let

$$\ell = \frac{b-a}{2}.$$

Then the interval

$$(a, b) = (m - \ell, m + \ell).$$

So, if we want to move this interval to $(-1, 1)$, we take $t \in (m - \ell, m + \ell)$ and map it to

$$t \mapsto \frac{t-m}{\ell} = x.$$

We see that $m \mapsto 0$ and the endpoints

$$m - \ell \mapsto \frac{m - \ell - m}{\ell} = -1, \quad m + \ell \mapsto \frac{m + \ell - m}{\ell} = 1.$$

It is a linear map, so everything in between maps to everything in between -1 and 1. So we have a bijection between (a, b) and $(-1, 1)$. If we want to go from $(-1, 1)$ to (a, b) then we send

$$x \in (-1, 1) \mapsto \ell x + m = t.$$

Since we know about the Legendre polynomials, P_n, on $(-1, 1)$ since $t \mapsto \frac{t-m}{\ell} = x$ sends (a, b) to $(-1, 1)$,

$$P_n\left(\frac{t-m}{\ell}\right) \qquad \text{are orthogonal on } (a, b).$$

To see this, just compute

$$\int_a^b P_n\left(\frac{t-m}{\ell}\right) P_k\left(\frac{t-m}{\ell}\right) dt = \int_{-1}^{1} P_n(x) P_k(x) dx = 0 \text{ if } n \neq k.$$

We have simply used substitution in the integral with $x = \frac{t-m}{\ell}$. So, these modified Legendre polynomials are orthogonal on (a, b). Moreover

$$\int_a^b P_n^2\left(\frac{t-m}{\ell}\right) dt = \int_{-1}^{1} P_n^2(x) dx = ||P_n||^2 = \frac{2}{2n+1}.$$

So, we simply expand the function f using this version of the Legendre polynomials. Let

$$c_n = \frac{\int_a^b f(t) P_n\left(\frac{t-m}{\ell}\right) dt}{||P_n||^2}.$$

The polynomial we seek is

$$P(t) = \sum_{n=0}^{5} c_n P_n\left(\frac{t-m}{\ell}\right).$$

1.3.1. *Weighted $\mathcal{L}^2$ on $\mathbb{R}$ with weight e^{-x^2}.* Find the polynomial of at most degree 4 which minimizes

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-x^2} dx.$$

We know that the Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with the weight function e^{-x^2}. We see that same weight function in the integral. Therefore, we can rely on the theory of the Hermite polynomials! Consequently, we define

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(x) e^{-x^2} dx}{||H_n||^2},$$

where

$$||H_n||^2 = \int_{\mathbb{R}} H_n^2(x) e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{4} c_n H_n(x).$$

Some variations on this theme are created by changing the weight function. For example, consider the problem: find the polynomial of at most degree 6 which minimizes

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-2x^2} dx.$$

This is not the correct weight function for H_n. However, we can make it so. The correct weight function for $H_n(x)$ is e^{-x^2}. So, if the exponential has $2x^2 = (\sqrt{2}x)^2$, then we should change the variable in H_n as well. We will then have, via the substitution $t = \sqrt{2}x$,

$$\int_{\mathbb{R}} H_n(\sqrt{2}x) H_m(\sqrt{2}x) e^{-2x^2} dx = \int_{\mathbb{R}} H_n(t) H_m(t) e^{-t^2} \frac{dt}{\sqrt{2}} = 0, \quad n \neq m.$$

Moreover, the norm squared is now

$$\int_{\mathbb{R}} H_n^2(t) e^{-t^2} \frac{dt}{\sqrt{2}} = \frac{||H_n||^2}{\sqrt{2}} = \frac{2^n n! \sqrt{\pi}}{\sqrt{2}}.$$

Consequently, the function $H_n(\sqrt{2}x)$ are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-2x^2}. We have computed the norms squared above. The coefficients are therefore

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(\sqrt{2}x) e^{-2x^2} dx}{2^n n! \sqrt{\pi}/\sqrt{2}}.$$

The polynomial is

$$P(x) = \sum_{n=0}^{6} c_n H_n(\sqrt{2}x).$$

1.3.2. *Weighted $\mathcal{L}^2$ on $(0, \infty)$ with weight $x^\alpha e^{-x}$.* This is rather unlikely to occur, because the Laguerre polynomials are rather scary, but it is possible. So, best that you are prepared for this eventuality. In this case, we know that the Laguerre polynomials are an orthogonal basis for this Hilbert space. So, if we are asked, for example, find the polynomial of at most degree 7 which minimizes

$$\int_0^\infty |f(x) - P(x)|^2 x^\alpha e^{-x} dx,$$

then we should define

$$c_n = \frac{\int_0^\infty f(x) L_n^\alpha(x) x^\alpha e^{-x} dx}{||L_n^\alpha||^2}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{7} c_n L_n^\alpha(x).$$

Variations on this theme? That is virtually unimaginable.

1.3.3. *Other functions and considerations.* We could ask the same type of question looking for coefficients of $\sin(nx)$ or $\cos(nx)$, for say $n = 0, 1, 2, 3, .., N$. Here, one uses the fact that those functions also yield orthogonal basis for $\mathcal{L}^2$ on bounded intervals. That is the name of the game: using the first N elements of an orthogonal basis for the $\mathcal{L}^2$ space under consideration.

You may wonder why when it says *at most degree* N we always find *all* the coefficients, c_0, c_1, $\ldots c_N$. That is because this is *better* then stopping at say c_{N-1}. Find them all. It could turn out that some of these end up being zero, so the polynomial has degree lower than N. The only way to know that is to check the calculation of all the c's, OR to know that certain coefficients will vanish due to evenness or oddness of functions, things of that nature. So, don't toss out any of the coefficients unless you are sure they vanish. Collect them all, like Pokemon!