FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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It is time to prepare for the exam! We will start by considering the other possible scenarios for “best approximation type problems.”

1.1. **Weighted $\mathcal{L}^2$ on $\mathbb{R}$ with weight similar to e^{-x^2}.** Find the polynomial of at most degree 4 which minimizes

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-x^2} dx.$$

We know that the Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with the weight function e^{-x^2}. We see that same weight function in the integral. Therefore, we can rely on the theory of the Hermite polynomials! Consequently, we define

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(x) e^{-x^2} dx}{||H_n||^2},$$

where

$$||H_n||^2 = \int_{\mathbb{R}} H_n^2(x) e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{4} c_n H_n(x).$$

Some variations on this theme are created by changing the weight function. For example, consider the problem: find the polynomial of at most degree 6 which minimizes

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-2x^2} dx.$$

This is not the correct weight function for H_n. However, we can make it so. The correct weight function for $H_n(x)$ is e^{-x^2}. So, if the exponential has $2x^2 = (\sqrt{2}x)^2$, then we should change the variable in H_n as well. We will then have, via the substitution $t = \sqrt{2}x$,

$$\int_{\mathbb{R}} H_n(\sqrt{2}x) H_m(\sqrt{2}x) e^{-2x^2} dx = \int_{\mathbb{R}} H_n(t) H_m(t) e^{-t^2} \frac{dt}{\sqrt{2}} = 0, \quad n \neq m.$$

Moreover, the norm squared is now

$$\int_{\mathbb{R}} H_n^2(t) e^{-t^2} \frac{dt}{\sqrt{2}} = \frac{||H_n||^2}{\sqrt{2}} = \frac{2^n n! \sqrt{\pi}}{\sqrt{2}}.$$

Consequently, the function $H_n(\sqrt{2}x)$ are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-2x^2}. We have computed the norms squared above. The coefficients are therefore

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(\sqrt{2}x) e^{-2x^2} dx}{2^n n! \sqrt{\pi}/\sqrt{2}}.$$

The polynomial is

$$P(x) = \sum_{n=0}^{6} c_n H_n(\sqrt{2}x).$$

1.2. **Weighted $\mathcal{L}^2$ on $(0, \infty)$ with weight $x^\alpha e^{-x}$.** This is rather unlikely to occur, because the Laguerre polynomials are rather scary, but it is possible. So, best that you are prepared for this eventuality. In this case, we know that the Laguerre polynomials are an orthogonal basis for this Hilbert space. So, if we are asked, for example, find the polynomial of at most degree 7 which minimizes

$$\int_0^\infty |f(x) - P(x)|^2 x^\alpha e^{-x} dx,$$

then we should define

$$c_n = \frac{\int_0^\infty f(x) L_n^\alpha(x) x^\alpha e^{-x} dx}{||L_n^\alpha||^2}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{7} c_n L_n^\alpha(x).$$

Variations on this theme? That is virtually unimaginable.

1.3. **Other best approximations.** We could ask the same type of question looking for coefficients of $\sin(nx)$ or $\cos(nx)$, for say $n = 0, 1, 2, 3, .., N$. Here, one uses the fact that those functions also yield orthogonal basis for $\mathcal{L}^2$ on bounded intervals. That is the name of the game: using the first N elements of an orthogonal basis for the $\mathcal{L}^2$ space under consideration.

You may wonder why when it says *at most degree* N we always find *all* the coefficients, c_0, c_1, $\ldots c_N$. That is because this is *better* then stopping at say c_{N-1}. Find them all. It could turn out that some of these end up being zero, so the polynomial has degree lower than N. The only way to know that is to check the calculation of all the c's, OR to know that certain coefficients will vanish due to evenness or oddness of functions, things of that nature. So, don't toss out any of the coefficients unless you are sure they vanish. Collect them all, like Pokemon!

1.4. **Theory Items: The Cliff Notes Version.**

Theorem 1 (Pointwise Convergence of Fourier Series)**.** *Let f be a 2π periodic function. Assume that f is piecewise continuous on $\mathbb{R}$, and that for every $x \in \mathbb{R}$, the left and right limits of both f and f' exist at x, and these are finite. Let*

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Then

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right), \quad \forall x \in \mathbb{R}.$$

Proof:

(1) Fix the point $x \in \mathbb{R}$.
(2) Expand the series.

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy e^{inx}.$$

(3) Change variable of integration to $t = y - x$. Use periodicity to say that the limits of integration don't change, so we have

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x)\frac{1}{2\pi}\sum_{-N}^{N} e^{int}dt.$$

(4) Define the N^{th} Dirichlet kernel:

$$D_N(t) = \frac{1}{2\pi}\sum_{-N}^{N} e^{int}.$$

It is even, integral from $\pm\pi$ to 0 is $\frac{1}{2}$. It is also equal to

$$D_N(t) = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}.$$

(5) In terms of D_N now

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x)D_N(t)dt.$$

(6) Use the fact about the integral of D_N to write

$$\frac{1}{2}f(x_-) = \int_{-\pi}^{0} D_N(t)dtf(x_-), \quad \frac{1}{2}f(x_+) = \frac{1}{2} = \int_{0}^{\pi} D_N(t)dtf(x_+).$$

(7) Show that now you just need to estimate:

$$\left|\int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-))dt + \int_{0}^{\pi} D_N(t)(f(t+x) - f(x_+))dt\right|.$$

(8) Insert the alternate expression for D_N,

fsest

$$\left| \int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_-))dt + \int_0^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_+))dt \right|. \tag{1.1}$$

(9) Define a new function

$$g(t) = \begin{cases} \frac{f(t+x)-f(x_-)}{1-e^{it}}, & t<0 \\ \frac{f(t+x)-f(x_+)}{1-e^{it}}, & t>0 \end{cases}$$

Show that g is piecewise continuous and piecewise differentiable, because

$$\lim_{t\to 0_-} \frac{f(t+x)-f(x_-)}{1-e^{it}} = \lim_{t\to 0_-} \frac{t(f(t+x)-f(x_-))}{t(1-e^{it})} = \frac{f'(x_-)}{-ie^{i0}} = if'(x_-)$$

and

$$\lim_{t\to 0_+} \frac{f(t+x)-f(x_-)}{1-e^{it}} = if'(x_+).$$

(10) Use Bessel's inequality for the Fourier coefficients of g to prove that (fsest 1.1) tends to zero as $N \to \infty$.

These are the main steps, learn them and practice filling in the gaps

Theorem 2 (Fourier coefficients of function and derivative)**.** *This time in Swedish for fun! Låt f vara en 2π-periodisk funktion med $f \in C^2(\mathbb{R})$. Sedan Fourierkoefficienterna c_n av f och Fourierkoefficienterna c_n' av f' uppfyller*

$$c_n' = inc_n.$$

Proof: Use the definition of the Fourier series and coefficients of f and f' and integrate by parts.

Theorem 3 (Big bad convolution approximation)**.** *Let $g \in L^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} g(x)dx = 1.$$

Define

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_0^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\varepsilon > 0$,

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon\to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

(1) This reduces to showing:

$$\lim_{\varepsilon \to 0} \int_{-\infty} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy = 0$$

and also

$$\lim_{\varepsilon \to 0} \int_{0}^{\infty} f(x-y) g_\varepsilon(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

(2) Choose one of these, saying the argument works in the same way for both. Make sure you understand why that is true!

(3) I take the first one. Observe that for $z = \varepsilon y$, we get

$$\int_{-\infty}^{0} g(y) dy = \int_{-\infty}^{0} g(z/\varepsilon) \frac{dz}{\varepsilon} = \int_{-\infty}^{0} g_\varepsilon(z) dz.$$

(4) Re-write the second term using $g_\varepsilon(y)$ because $f(x+)$ is a constant, so you are estimating

$$\int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy.$$

(5) Split the integral into two terms:

$$\int_{-\infty}^{y_0} + \int_{y_0}^{0}.$$

Remember that $y_0 < 0$.

(6) START WITH THE SECOND OF THESE. Look at

$$\int_{y_0}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy.$$

Use the definition of $f(x+)$ and the fact that $y < 0$ so $x - y > y$ to argue that there exists $y_0 < 0$ such that $|f(x-y) - f(x+)|$ can be made as small as you want. Then estimate

$$\left| \int_{y_0}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right| \leq \text{small} \int_{y_0}^{0} |g_\varepsilon(y)| dy \leq \text{small} \int_{\mathbb{R}} |g_\varepsilon(y)| dy = \text{small} ||g||_{\mathcal{L}^1}.$$

The $\mathcal{L}^1$ norm of g is finite by assumption, so this can be made as small as desired.

(7) NOW THAT y_0 is FIXED, look at the integral

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+)) g_\varepsilon(y) dy \right|.$$

If f is bounded by $M > 0$, then $|f(x-y) - f(x+)| \leq 2M$. Estimating:

$$2M \left| \int_{-\infty}^{y_0} g_\varepsilon(y) dy \right| = 2M \int_{-\infty}^{y_0/\varepsilon} |g(z)| dz.$$

Since $y_0 < 0$, as $\varepsilon \to 0$, $y_0/\varepsilon \to -\infty$. Tail of convergent integral, so can take $\varepsilon > 0$ but small to make $\int_{-\infty}^{y_0/\varepsilon} |g(z)| dz$ as small as we want.

(8) If g vanishes outside of a bounded interval, take $\varepsilon > 0$ so small that g vanishes for $z \in (-\infty, y_0/\varepsilon)$. Then you also make this term small (zero in fact).

Theorem 4 (FIT)**.** *For $f \in \mathcal{L}^2(\mathbb{R})$,*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi.$$

Theorem 5 (Plancharel)**.** *For f and g in $\mathcal{L}^2(\mathbb{R})$,*

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x)\overline{\hat{g}(x)} dx = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle.$$

Proof:

(1) Start with the left side and use the FIT on f, to write

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \overline{g(x)} d\xi dx.$$

(2) Move the complex conjugate to engulf the $e^{ix\xi}$,

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \overline{g(x) e^{-ix\xi}} d\xi dx.$$

(3) Swap the order of integration and integrate x first:

$$\int_{\mathbb{R}} \hat{f}(\xi) \overline{g(x) e^{-ix\xi}} dx = \hat{f}(\xi) \overline{\hat{g}(\xi)}.$$

Put it back in:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle.$$

Theorem 6 (Sampling)**.** *Let $f \in L^2(\mathbb{R})$. We take the definition of the Fourier transform of f to be*

$$\int_{\mathbb{R}} e^{-ix\xi} f(x) dx,$$

and we then assume that there is $L > 0$ so that $\hat{f}(\xi) = 0 \ \forall \xi \in \mathbb{R}$ with $|\xi| > L$. Then:

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

Proof:

(1) Because $\hat{f}$ is zero outside the interval $[-L, L]$, expand it in a Fourier series:

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx.$$

(2) Use the FIT and keep in mind that $\hat{f}$ is zero outside $[-L, L]$:

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixt} \hat{f}(x) dx = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \hat{f}(x) dx.$$

(3) Substitute Fourier expansion of $\hat{f}$ into the integral:

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} c_n e^{in\pi x/L} dx.$$

(4) Investigate the coefficients (keep in mind that $\hat{f}(x) = 0$ for $|x| > L$) and use the FIT:

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx = \frac{1}{2L}\int_{\mathbb{R}} e^{ix(-n\pi/L)}\hat{f}(x)dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

(5) Substitute into:

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty}\frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L}dx$$

(6) Swap n and $-n$ because that's just changing the order of summation which doesn't change a thing.

$$f(t) = \frac{1}{2L}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\int_{-L}^{L} e^{x(it-in\pi/L)}dx.$$

(7) Compute

$$\int_{-L}^{L} e^{x(it-in\pi/L)}dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)}\sin(Lt-n\pi).$$

Reduce to obtain (use fact that sine is odd):

$$\frac{\sin(Lt-n\pi)}{Lt-n\pi} = \frac{-\sin(n\pi-Lt)}{Lt-n\pi} = \frac{\sin(n\pi-Lt}{n\pi-Lt}.$$

Theorem 7 (Three equivalent conditions to be an ONB in Hilbert space)**.** *Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum, H. Följande tre är ekvivalenta:*

$$(1) \quad f \in H \text{ och } \langle f, \phi_n\rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}}\langle f, \phi_n\rangle\phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n\in\mathbb{N}}|\langle f, \phi_n\rangle|^2.$$

Proof:

(1) Show (1) $\implies$ (2) $\implies$ (3) $\implies$ (1). Start by assuming (1) holds.
(2) Observe that

$$\sum_{n\in\mathbb{N}}|\langle f, \phi_n\rangle|^2 \le ||f||^2 < \infty.$$

(3) Use this and the Pythagorean Theorem to show that if

$$g_N := \sum_{n=0}^{N}\langle f, \phi_n\rangle\phi_n,$$

then $\{g_N\}$ is a Cauchy sequence.
(4) Hilbert spaces are complete so $g_N \to g \in H$.
(5) Show that $\langle f - g, \phi_n\rangle = 0$ for all n. Since (1) is true, this means $f = g$.
(6) Next you assume (2) holds. Use the Pythagorean Theorem to obtain (3).
(7) Next you assume (3) holds. If $\langle f, \phi_n\rangle = 0$ for all n, then by (3), $||f||^2 = 0$ so $f = 0$.

Theorem 8 (Best Approximation). *Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara en ortonormal mängd i ett Hilbert-rum, H. Om $f \in H$,*

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \le ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

och $=$ gäller $\iff$ $c_n = \langle f, \phi_n\rangle$ gäller $\forall n \in \mathbb{N}$.

Proof:

(1) Define
$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$
and
$$\varphi := \sum c_n \phi_n.$$

(2) Then,
$$||f-\varphi||^2 = ||f-g+g-\varphi||^2 = ||f-g||^2 + ||g-\varphi||^2 + 2\Re\langle f-g, g-\varphi\rangle.$$

(3) Stay calm and carry on, using the properties of the scalar product to show that
$$\langle f-g, g-\varphi\rangle = 0.$$

(4) Thus
$$||f-\varphi||^2 = ||f-g+g-\varphi||^2 = ||f-g||^2 + ||g-\varphi||^2 \le ||f-g||^2,$$
with equality if and only if $||g-\varphi||^2 = 0$. Use the Pythagorean Theorem to show that
$$||g-\varphi||^2 = \sum \left|\widehat{f_n} - c_n\right|^2,$$
so this vanishes iff $c_n = \widehat{f_n}$ for all n.

Theorem 9 (SLPs). *Låt f och g vara egenfunktioner till ett regulärt SLP i intervallet $[a,b]$ med $w \equiv 1$. Låt λ vara egenvärden till f och μ vara dess till g. Sedan gäller:*

(1) $\lambda \in \mathbb{R}$ och $\mu \in \mathbb{R}$;
(2) Om $\lambda \ne \mu$, gäller:
$$\int_a^b f(x)\overline{g(x)}dx = 0.$$

Proof:

(1) The scalar product here is
$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

(2) Self-adjointness in the definition of regular SLP says
$$\langle f, Lg\rangle = \langle Lf, g\rangle.$$

(3) Use this, properties of the scalar product, and the eigenvalue equations for f and g to obtain the results. Just follow your nose.

Theorem 10 (Bessel generating function)**.** *For all x and for all $z \neq 0$, the Bessel functions, J_n satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}.$$

(1) Write out TWO TAYLOR SERIES:

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

(2) Multiply the series together

bessel1

$$e^{xz/2} e^{-x/(2z)} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k \geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}. \tag{1.2}$$

(3) You have TWO independent variables. Change ONE of them. Let $n = j-k$. So you use the two independent variables n and k. Now n goes from $-\infty$ to ∞. Also, $j+k = n+2k$, and $j = n+k$ Thus:

$$e^{xz/2} e^{-x/(2z)} = \sum_{n \in \mathbb{Z}} \sum_{k \geq 0} (-1)^k z^n \frac{(x/2)^{n+2k}}{k!(n+k)!}.$$

Pull the z^n in front and recall that $(n+k)! = \Gamma(n+k+1)$ to have

$$e^{xz/2} e^{-x/(2z)} = \sum_{n \in Z} z^n \sum_{k \geq 0} (-1)^k \frac{(x/2)^{n+2k}}{k!\Gamma(n+k+1)} = \sum_{n \in \mathbb{Z}} z^n J_n(x).$$

Theorem 11 (Hermite orthogonality)**.** *The Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Recall here that*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

and so the statement is that

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = 0, \quad n \neq m.$$

Proof:

(1) Assume without loss of generality that $n > m$. Write using the definition of H_n and H_m,

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = (-1)^n \int_{\mathbb{R}} \frac{d^n}{dx^n} e^{-x^2} H_m(x) dx.$$

(2) Use integration by parts n times. The boundary terms vanish because

$$\frac{d^j}{dx^j} e^{-x^2} = p_j(x) e^{-x^2}$$

for all j, where p_j is some polynomial of degree j. Thus the boundary terms are of the form polynomial times e^{-x^2} and the decay of the Gaussian makes everything vanish at $\pm\infty$. After integrating by parts n times get:

$$\int_{\mathbb{R}} e^{-x^2} \frac{d^n}{dx^n} H_m(x) dx.$$

(3) H_m is a polynomial of degree $m < n$. So if you differentiate it n times the result is zero. Hence the integrand is zero and that whole hot mess is zero.

Theorem 12 (Generating for Hermite)**.** *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

satisfy

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Proof:

(1) Define a function:

$$e^{-(x-z)^2} = e^{-x^2 + 2xz - z^2}.$$

(2) Do a Taylor series expansion of this function at the point $z = 0$:

$$e^{-(x-z)^2} = \sum_{n \geq 0} a_n z^n,$$

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} e^{-(x-z)^2}, \qquad \text{evaluated at } z = 0.$$

(3) Compute the coefficients using the chain rule. The exponent is $-(x-z)^2$. Let $u = x - z$. Then $\frac{d}{dz} u = -1$ so

$$\frac{d^n}{dz^n} e^{-(x-z)^2} \bigg|_{z=0} = (-1)^n \frac{d^n}{du^n} e^{-u^2} \bigg|_{z=0}.$$

Since $u = x - z$, when $z = 0$, $u = x$, so

$$= (-1)^n \frac{d^n}{du^n} e^{-u^2} \bigg|_{u=x}.$$

This is the same as

$$(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

(4) Insert the coefficients into the Taylor series:

$$e^{-(x-z)^2} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} e^{-x^2} z^n.$$

(5) Multiply both sides by e^{x^2}:

$$e^{x^2} e^{-(x-z)^2} = e^{2xz - z^2} = \sum_{n \geq 0} \frac{(-1)^n}{n!} e^{x^2} \frac{d^n}{dx^n} e^{-x^2} z^n = \sum_{n \geq 0} \frac{z^n}{n!} H_n(x).$$