FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2018.03.9

We continue preparing for the exam!

1.1. **Summing series using Fourier series.** We are asked to compute:

$$\sum_1^\infty \frac{1}{9+n^2}.$$

We are given the hint to expand e^{3x} in a Fourier series on $(-\pi, \pi)$. So really, we extend the function to be 2π periodic on $\mathbb{R}$. We compute the Fourier coefficients

$$\int_{-\pi}^{\pi} e^{3x} e^{-inx} dx = \left. \frac{e^{x(3-in)}}{3-in} \right|_{x=-\pi}^{x=\pi} = \frac{e^{3\pi} e^{-in\pi}}{3-in} - \frac{e^{-3\pi} e^{in\pi}}{3-in} = (-1)^n \frac{2\sinh(3\pi)}{3-in}.$$

Hence, the Fourier coefficients are

$$\frac{1}{2\pi}(-1)^n \frac{2\sinh(3\pi)}{3-in},$$

and the Fourier series is

$$\sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(3\pi)}{\pi(3-in)} e^{inx}.$$

We pull out some constant stuff,

$$\frac{\sinh(3\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{3-in}.$$

How can we get a series of $\frac{1}{9+n^2}$? We need that pesky $(-1)^n$ to go away. It will go away if $x = \pi$ because then $e^{in\pi} = (-1)^n$ so $(-1)^n e^{in\pi} = 1$. Moreover once that is the case, observe that pairing positive and negative n's, we get

$$n > 0, \quad \frac{1}{3+in} + \frac{1}{3-in} = \frac{3-in+3+in}{(3+in)(3-in)} = \frac{6}{9+n^2}.$$

The term $n = 0$ is $1/3$. So, for $x = \pi$ the series is

$$\frac{\sinh(3\pi)}{\pi} \left(\frac{1}{3} + \sum_{n \geq 1} \frac{6}{9+n^2} \right).$$

Now, we use the Theorem on the pointwise convergence of Fourier series. Our function is extended to be 2π periodic. When we approach the point $x = \pi$ from the left, the function is tending to $e^{3\pi}$. However, since we extend it to be 2π periodic, if we approach from the right, that is for $x > \pi$, the limit is $e^{-3\pi}$ (the same as if we approach the point $-\pi$ from the right because of periodicity). Therefore, the theorem says that the Fourier series

$$\frac{\sinh(3\pi)}{\pi}\left(\frac{1}{3} + \sum_{n\geq 1}\frac{6}{9+n^2}\right) \text{ converges to } \frac{e^{3\pi}+e^{-3\pi}}{2} = \cosh(3\pi).$$

We simply re-arrange:

$$\frac{\pi\cosh(3\pi)}{\sinh(3\pi)} = \frac{1}{3} + 6\sum_{n\geq 1}\frac{1}{9+n^2} \implies \sum_{n\geq 1}\frac{1}{9+n^2} = \left(\frac{\pi\cosh(3\pi)}{\sinh(3\pi)} - \frac{1}{3}\right)\frac{1}{6}.$$

1.2. **Best approximation.** Minimize:

$$\int_{-2}^{2} |\cos(2\pi x) + P(x)|^2 dx, \qquad \text{degree of } P \text{ is at most } 2.$$

This looks weird because it is $+$ not $-$. Very sneaky indeed. However, we just do for the usual case, then use $-P(x)$ instead. So, we see that the interval is a nice finite interval, and we can use the Legendre polynomials, with a slight modification. Consider

$$t = \frac{x}{2}.$$

Then we have

$$\int_{-2}^{2} P_n(x/2)P_m(x/2)dx = \int_{-1}^{1} P_n(t)P_m(t)2dt = \begin{cases} 0 & n \neq m \\ \frac{4}{2n+1} & n = m \end{cases}$$

Hence the polynomials $\{P_n(x/2)\}_{n\geq 0}$ are orthogonal on $\mathcal{L}^2$ on the interval $(-2, 2)$. We use these to approximate the function:

$$c_j = \frac{\int_{-2}^{2} P_j(x/2)\cos(2\pi x)dx}{4/(2j+1)}.$$

The polynomial which best approximates $\cos(2\pi x)$ is

$$\sum_{0}^{2} c_j P_j(x/2).$$

So the one we seek above is

$$P(x) = -\sum_{0}^{2} c_n P_n(x/2).$$

1.3. **Solving PDE on finite interval.** We should solve:

$$\begin{cases} u_t - ku_{xx} = 30x & x \in (0,1), \quad t > 0 \\ u(x,0) = g(x) & x \in (0,1) \\ u(0,t) = 0 = u(1,t) \end{cases}$$

We see that the PDE is inhomogeneous with a *time independent* inhomogeneity. So, we search for a steady state solution. We need

$$-kf''(x) = 30x \implies -kf'(x) = 15x^2 + c \implies -kf(x) = 5x^3 + cx + b.$$

Hence,

$$f(x) = -\frac{5x^3}{k} - \frac{cx}{k} - \frac{b}{k}.$$

To satisfy the boundary conditions, we need

$$f(0) = -\frac{b}{k} = 0 \implies b = 0.$$

We also need

$$f(1) = -\frac{5}{k} - \frac{c}{k} = 0 \implies c = -5.$$

So,

$$f(x) = -\frac{5x^3}{k} + \frac{5x}{k}.$$

We next solve the problem:

$$\begin{cases} u_t - ku_{xx} = 0 & x \in (0,1), \quad t > 0 \\ u(x,0) = g(x) - f(x) & x \in (0,1) \\ u(0,t) = 0 = u(1,t) \end{cases}$$

We do this using separation of variables. Write $u = X(x)T(t)$. The PDE is then

$$T'X - X''T = 0 \implies \frac{T'}{T} = \frac{X''}{X}.$$

Both sides are constant. Call it λ. So, we have

$$X'' = \lambda X, \quad X(0) = X(1) = 0.$$

We must consider $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. I leave it as an exercise to verify that there are no non-zero solutions for $\lambda \geq 0$. For $\lambda < 0$ we have solutions, up to constant factors,

$$X_n(x) = \sin(n\pi x), \quad \lambda_n = -n^2\pi^2.$$

This gives us the equation for the partner function, T_n,

$$T_n'(t) = \lambda_n T_n \implies T_n(t) = e^{-n^2\pi^2 t} \text{ up to constant factors.}$$

Since we are solving the homogeneous PDE, we smash the solutions together into a series,

$$u(x,t) = \sum_{n \geq 1} c_n T_n(t) X_n(x).$$

Since we want

$$u(x,0) = \sum_{n \geq 1} c_n X_n(x) = g(x) - f(x),$$

this tells us that

$$c_n = \frac{\int_0^1 (g(x) - f(x)) X_n(x) dx}{\int_0^1 X_n^2(x) dx}.$$

The full solution is

$$f(x) + u(x,t).$$

1.4. **PDE on $\mathbb{R}$.** We should solve the transport equation:

$$\begin{cases} u_t + cu_x = g(x,t) & x \in \mathbb{R}, t > 0 \\ u(x,0) = \varphi(x) \end{cases}$$

We hit the PDE with the Fourier transform in the x variable, because the PDE is on $\mathbb{R}$:

$$\partial_t \hat{u}(\xi,t) + ci\xi\hat{u}(\xi,t) = \hat{g}(\xi,t).$$

Above we have used BETA 13.2 F.10. Next, we use another formula from BETA, for first order ODEs, to solve the ODE in t. The formula is 9.1.3, which says that the solution is

$$\hat{u}(\xi,t) = e^{-ci\xi t}\left(\int_0^t e^{ic\xi s}\hat{g}(\xi,s)ds + C(\xi)\right).$$

To have the correct initial data,

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi) \implies C(\xi) = \hat{\varphi}(\xi).$$

So, we have found

$$\hat{u}(\xi,t) = e^{-ci\xi t}\left(\int_0^t e^{ic\xi s}\hat{g}(\xi,s)ds + \hat{\varphi}(\xi)\right) = e^{-ic\xi t}\hat{\varphi}(\xi) + \int_0^t e^{-ic\xi(t-s)}\hat{g}(\xi,s)ds.$$

To go backwards, we use BETA 13.2.F7,

$$u(x,t) = \varphi(x-ct) + \int_0^t g(x-c(t-s),s)ds.$$

1.5. **PDE on $\mathbb{R}^+$.** Solve:

$$\begin{cases} u_x + cu_t + u = 0 & x > 0, \quad t > 0 \\ u(0,t) = g(t) \\ u(x,0) = 0 \end{cases}$$

We use the Laplace transform in the t variable, because we have the nice condition that $u(x,0) = 0$. So, we have

$$\partial_x \tilde{u}(x,z) + cz\tilde{u}(x,z) + \tilde{u}(x,z) = 0.$$

This is a nice ODE for $\tilde{u}$,

$$\partial_x \tilde{u}(x,z) = -(cz+1)\tilde{u}(x,z) \implies \tilde{u}(x,z) = a(z)e^{-(cz+1)x}.$$

The BC tells us that

$$\tilde{u}(0,z) = \tilde{g}(z) = a(z).$$

So, we have found

$$\tilde{u}(x,z) = \tilde{g}(z)e^{(-cz+1)x} = e^{-czx}\tilde{g}(z)e^{-x}.$$

To undo the Laplace transform we use BETA 13.5 L4:

$$u(x,t) = g(t-cx)\Theta(t-cx)e^{-x}.$$

Above, Θ is the heavyside function.

1.6. **The last three theory items: short proofs.**

Theorem 1 (Bessel generating function)**.** *For all x and for all $z \neq 0$, the Bessel functions, J_n satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}.$$

(1) Write out TWO TAYLOR SERIES:

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

(2) Multiply the series together

bessel1

$$e^{xz/2} e^{-x/(2z)} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k \geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}. \tag{1.1}$$

(3) You have TWO independent variables. Change ONE of them. Let $n = j-k$. So you use the two independent variables n and k. Now n goes from $-\infty$ to ∞. Also, $j + k = n + 2k$, and $j = n + k$ Thus:

$$e^{xz/2} e^{-x/(2z)} = \sum_{n \in \mathbb{Z}} \sum_{k \geq 0} (-1)^k z^n \frac{(x/2)^{n+2k}}{k!(n+k)!}.$$

Pull the z^n in front and recall that $(n + k)! = \Gamma(n + k + 1)$ to have

$$e^{xz/2} e^{-x/(2z)} = \sum_{n \in Z} z^n \sum_{k \geq 0} (-1)^k \frac{(x/2)^{n+2k}}{k!\Gamma(n+k+1)} = \sum_{n \in \mathbb{Z}} z^n J_n(x).$$

Theorem 2 (Hermite orthogonality)**.** *The Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Recall here that*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

and so the statement is that

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = 0, \quad n \neq m.$$

Proof:

(1) Assume without loss of generality that $n > m$. Write using the definition of H_n and H_m,

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = (-1)^n \int_{\mathbb{R}} \frac{d^n}{dx^n} e^{-x^2} H_m(x) dx.$$

(2) Use integration by parts n times. The boundary terms vanish because

$$\frac{d^j}{dx^j} e^{-x^2} = p_j(x) e^{-x^2}$$

for all j, where p_j is some polynomial of degree j. Thus the boundary terms are of the form polynomial times e^{-x^2} and the decay of the Gaussian makes everything vanish at $\pm\infty$. After integrating by parts n times get:

$$\int_{\mathbb{R}} e^{-x^2} \frac{d^n}{dx^n} H_m(x) dx.$$

(3) H_m is a polynomial of degree $m < n$. So if you differentiate it n times the result is zero. Hence the integrand is zero and that whole hot mess is zero.

Theorem 3 (Generating for Hermite)**.** *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

satisfy

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Proof:

(1) Define a function:

$$e^{-(x-z)^2} = e^{-x^2 + 2xz - z^2}.$$

(2) Do a Taylor series expansion of this function at the point $z = 0$:

$$e^{-(x-z)^2} = \sum_{n \geq 0} a_n z^n,$$

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} e^{-(x-z)^2}, \qquad \text{evaluated at } z = 0.$$

(3) Compute the coefficients using the chain rule. The exponent is $-(x-z)^2$. Let $u = x - z$. Then $\frac{d}{dz} u = -1$ so

$$\left. \frac{d^n}{dz^n} e^{-(x-z)^2} \right|_{z=0} = (-1)^n \left. \frac{d^n}{du^n} e^{-u^2} \right|_{z=0}.$$

Since $u = x - z$, when $z = 0$, $u = x$, so

$$= (-1)^n \left. \frac{d^n}{du^n} e^{-u^2} \right|_{u=x}.$$

This is the same as

$$(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

(4) Insert the coefficients into the Taylor series:

$$e^{-(x-z)^2} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} e^{-x^2} z^n.$$

(5) Multiply both sides by e^{x^2}:

$$e^{x^2} e^{-(x-z)^2} = e^{2xz-z^2} = \sum_{n\geq 0} \frac{(-1)^n}{n!} e^{x^2} \frac{d^n}{dx^n} e^{-x^2} z^n = \sum_{n\geq 0} \frac{z^n}{n!} H_n(x).$$