FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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What is this mysterious L^2 thing? It seems to have raised a few eyebrows. So, let's demystify it a bit. Let us fix a finite (not infinite) interval $[a, b]$. Then $L^2([a, b])$ is a complete normed vector space which has an inner product. Once we have fixed the interval, we may simply write L^2. The "vectors" in this vector space can be understood as functions.[1] Of course, you've memorized that the inner product of two elements of L^2 is

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

For this reasons, the functions in L^2 need to satisfy:

$$\langle f, f\rangle = \int_a^b f(x)\overline{f(x)}dx = \int_a^b |f|^2 < \infty.$$

This guarantees that the elements of L^2 all have definable norms,

$$||f|| = \sqrt{\langle f, f\rangle}.$$

So, as long as you can integrate $|f|^2$ over the interval $[a, b]$ and get something finite, $f \in L^2$. Although we don't necessarily need it right now, you may be happy to know that the inner product satisfies a Cauchy-Schwarz inequality,

$$|\langle f, g\rangle| \leq ||f||||g||.$$

Exercise 1. *Use the Cauchy-Schwarz inequality to prove that for any $f \in L^2$ on the interval $[-\pi, \pi]$, the Fourier coefficients,*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

[1]For those of you interested in the finer details, they are actually equivalence classes of measurable functions. However, since we can always select a function to represent its equivalence class, we're just going to think about the elements of L^2 as measurable functions. Measurable is a notion from measure theory, and you can just take it on faith that *everything* you've ever seen or heard of or could possibly dream up is measurable. So just don't worry about the measurable part.

satisfy

$$|c_n| \leq \frac{||f||}{\sqrt{2\pi}}.$$

What are some examples? Well, any function f which is bounded on the interval will be an L^2 function. Let's make this official in what we'll call the standard estimate.

Proposition 1 (The standard estimate)**.** *Assume f is defined on some interval $[a, b]$. Assume that f satisfies a bound of the form $|f(x)| \leq M$ for $x \in [a, b]$. (We actually only need this for "almost every" x, but to make that precise, we need some Lebesgue measure theory). Then,*

$$\left| \int_a^b f(x)dx \right| \leq (b-a)M.$$

Proof: Standard estimate!

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx \leq \int_a^b M dx = M(b-a).$$

So, if f is bounded on an interval, then $|f|^2 \leq M^2$, is also bounded, hence the integral is bounded. Something like $f(x) = \frac{1}{x}$ will be problematic if the interval contains 0. However, even though $f(x) = x^{-1/3}$ blows up as $x \to 0$, it blows up slowly enough that

$$\int_{-\pi}^{\pi} |x^{-1/3}|^2 dx < \infty.$$

So, the function doesn't have to be bounded for the integral to be finite, but it also can't blow up too badly.

2. Bessel's Inequality (L^2 convergence of Fourier series)

Today we're going to investigate the issue of convergence of Fourier series. To move towards this question of convergence, we prove an important estimate known as the Bessel Inequality.

Theorem 2 (Bessel Inequality)**.** *Assume that f is 2π periodic and integrable on $[-\pi, \pi]$. Then the Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$ satisfy*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Proof: It's all about estimating. We want to show that for any N, we have

$$S_N := \sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Then, since it's true for all N, it's true as $N \to \infty$, so it's true for the whole sum over $n \in \mathbb{Z}$. Note that this is equivalent to showing that

$$0 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx - S_N \iff 0 \leq \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi S_N.$$

We have seen above that L^2 is a complete normed vector space. So, we can calculate the "distance" between elements of this vector space using the norm:

$$||f-g|| = \sqrt{\int_a^b |f(x)-g(x)|^2 dx}.$$

So, let us cook up a function who has L^2 norm equal to S_N (or maybe $2\pi S_N$) and whose distance to f we want to estimate. Of course, a natural guess is the partial Fourier series,

$$g_N := \sum_{-N}^{N} c_n e^{inx}.$$

Then, we compute

$$\begin{aligned}\int_{-\pi}^{\pi} |g_N(x)|^2 dx &= \int_{-\pi}^{\pi} g_N(x)\overline{g_N(x)} dx \\ &= \int_{-\pi}^{\pi} \sum_{n=-N}^{N} c_n e^{inx} \sum_{m=-N}^{N} \overline{c_m e^{imx}} dx \\ &= \sum_{n,m=-N}^{N} c_n \overline{c_m} \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx.\end{aligned}$$

These integrals are zero whenever $n \neq m$. Why? (Well, because we proved it...) So, we only get terms in the sum with $n = m$, hence we may use the same letter, and all the other terms are gone because they vanish. So,

$$||g_N||^2 = \sum_{n=-N}^{N} |c_n|^2 2\pi = 2\pi S_N.$$

Remember that when $n = m$ the scalar product $\langle e^{inx}, e^{imx} \rangle = 2\pi$. Super. So, we are now going to compare g_N and f.

For the sake of simplicity, we will drop the limits of integration, because all integrals are from $-\pi$ to π. We will also write f for $f(x)$ and drop the dx from the integral. To compare f to g_N, we start by looking at the L^2 distance between them:

$$\begin{aligned}\star = \int |f-g_N|^2 &= \int (f-g_N)\overline{f-g_N} \\ &= \int |f|^2 - g_N\overline{f} - f\overline{g_N} + |g_N|^2\end{aligned}$$

$$= \int |f|^2 - \sum_{n=-N}^{N} c_n \int \overline{f} e^{inx} - \sum_{n=-N}^{N} \overline{c_n} \int f e^{-inx} + \sum_{n,m=-N}^{N} c_n \overline{c_m} \int e^{inx} e^{-imx}.$$

Above, we used the linearity of the integral. Next, we use the definition of c_n,

$$c_n = \frac{1}{2\pi} \int f e^{-inx} \implies \overline{c_n} = \frac{1}{2\pi} \overline{\int f e^{-inx}} = \frac{1}{2\pi} \int \overline{f e^{-inx}} = \frac{1}{2\pi} \int \overline{f} e^{inx}.$$

So, we use this to simplify above,

$$\star = \int |f|^2 - \sum_{n=-N}^{N} c_n 2\pi \overline{c_n} - \sum_{n=-N}^{N} \overline{c_n} 2\pi c_n + \sum_{n,m=-N}^{N} c_n \overline{c_m} \int e^{inx} e^{-imx}.$$

The last term simplifies by noting that the integral vanishes whenever $n \neq m$. When $n = m$ the integral is 2π. So noting that $c_n \overline{c_n} = |c_n|^2$,

$$\star = \int |f|^2 - \sum_{n=-N}^{N} c_n 2\pi \overline{c_n} - \sum_{n=-N}^{N} \overline{c_n} 2\pi c_n + \sum_{n=-N}^{N} c_n \overline{c_n} 2\pi$$

$$= \int |f|^2 - 2\pi \sum_{n=-N}^{N} |c_n|^2.$$

What can we say about $\star$ from the beginning? By definition

$$\star = \int |f - g_N|^2 \geq 0,$$

because the integral of something non-negative is non-negative. So,

$$0 \leq \star = \int |f|^2 - 2\pi \sum_{n=-N}^{N} |c_n|^2 \implies 0 \leq \frac{1}{2\pi} \int |f|^2 - S_N \implies S_N \leq \frac{1}{2\pi} \int |f|^2.$$

This is what we wanted to show.

Corollary 3. *We have*

$$\sum_{n \in \mathbb{N}} |a_n|^2 + |b_n|^2 \leq 2 \sum_{n \in \mathbb{Z}} |c_n|^2,$$

and

$$\lim_{|n| \to \infty} \star_n = 0, \quad \star = a, b, \text{ or } c.$$

Exercise 2. *The proof is an exercise. First, use the previous exercises to express the a's and b's in terms of the c's. Next, what can you say about the terms of a non-negative, convergent series?*

3. Pointwise convergence of Fourier Series

By Bessel's inequality, we know that

$$\sum_{n \in \mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2.$$

Now, it's important to note that when the series of $|c_n|^2$ converges, then eventually $|c_n|^2 < 1$ so also $|c_n| < 1$. Then, $|c_n| > |c_n|^2$. So, just because the series of $|c_n|^2$ converges, the series with just c_n might not. For example,

$$\sum_{n \geq 1} \frac{1}{n^2} < \infty$$

whereas

$$\sum_{n \geq 1} \frac{1}{n} = \infty.$$

So Bessel's inequality doesn't tell us that the Fourier series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

always converges. This is a bit of a concern, because we want to use our method to solve PDEs. If our solution is one of these Fourier series, then we're up a creek without a paddle if that series doesn't converge to anything!

This is the motivation to investigate the subtle question of pointwise convergence of Fourier series. Although math is fun just for itself, here, we're always motivated by a desire to understand real, relevant, physical and chemical processes! (Like heat, waves, electromagnetism, quantum particles, chemical reactions, the hydrogen and other atoms, etc...)

Definition 4. We say that a function is piecewise $\mathcal{C}^k$ on a (possibly infinite) interval, I, if there is a discrete set, S of points in the interval (possibly empty set) such that f is $\mathcal{C}^k$ on $I \setminus S$. Moreover, we assume that the left and right limits of $f^{(j)}$ exist at all of the points in S, for all $j = 0, 1, \ldots, k$.

In case the notion of discrete set is unfamiliar, if the set S contains finitely many points, then it's a discrete set. If the interval $I = \mathbb{R}$, then both $\mathbb{Z}$ and $\mathbb{N}$ are discrete sets, but $\mathbb{Q}$ is not. To be perfectly precise, a set in $\mathbb{R}$ is discrete if it is countable, thus we may write it as $\{p_n\}_{n \in \mathbb{N}}$, and for each p_n there exists $\varepsilon_n > 0$ such that $|p_n - p_m| > \varepsilon_n \forall m \neq n$. That is, in the little interval $[p_n - \varepsilon_n/2, p_n + \varepsilon_n/2]$, the only point of our discrete set contained in that interval is p_n.

Examples of piecewise $\mathcal{C}^1$ functions are our periodically extended $|x|$, which is continuous on $\mathbb{R}$ but only piecewise $\mathcal{C}^1$. The periodically extended x is piecewise $\mathcal{C}^0$ and also piecewise $\mathcal{C}^1$. Actually, both of these guys are piecewise $\mathcal{C}^\infty$, because apart from the odd multiples of π, (and 0 for $|x|$) these functions are lovely and smooth.

Now we are going to prove the great big theorem about pointwise convergence of Fourier series.

Theorem 5. *Let f be a 2π periodic function. Assume that f is piecewise $\mathcal{C}^1$ on $\mathbb{R}$, where piecewise $\mathcal{C}^1$ is defined as above. Denote the left limit at x by $f(x_-)$ and the right limit by $f(x_+)$. Let*

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Then

$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} \left(f(x_-) + f(x_+) \right), \quad \forall x \in \mathbb{R}.$$

Proof: This is a big theorem, because it's got a lot of clever ideas in the proof. Smaller theorems can be proven by just "following your nose." So, to try to help with the proof, we're going to highlight the big ideas. To learn the proof, you can start by learning all the big ideas in the order in which they're used. Once you've got these down, then try to fill in the math steps starting at one idea, working to get to the next idea. The big ideas are like light posts guiding your way through the dark and spooky math.

To begin with, the result should hold for each and every point $x \in \mathbb{R}$. Idea 1: Fix a point $x \in \mathbb{R}$.

Next, as usual, we should use the definitions.

Idea 2: Expand the series using its definition.

So, we write

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx}.$$

Now, let's move that lonely e^{inx} inside the integral so it can get close to its friend, e^{-iny}. Then,

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny+inx} dy.$$

OBS that f on the right is not involved with x, but in the theorem we are trying to prove, we want to relate $S_N(x)$ to $f(x)$. How can we get an x inside the f?

Idea 3: Change the variable. Let $t = y - x$.

Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt.$$

Remember that very first fact we proved for periodic functions? It said that the integral of a periodic function of period P from any point a to $a+P$ is the same, no matter what a is. Here $P = 2\pi$. This leads to...

Idea 4: Use the Lemma on integrals of periodic functions to shift the integral

$$\int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt = \int_{-\pi}^{\pi} f(t+x) e^{-int} dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) e^{-int} dt = \int_{-\pi}^{\pi} f(t+x) \frac{1}{2\pi} \sum_{-N}^{N} e^{int} dt.$$

This is how we get to the Idea 4: Define the N^{th} Dirichlet kernel, $D_N(t)$, and investigate it by (1) collecting even and odd terms and (2) expressing it like a geometric series.

$$D_N(t) = \frac{1}{2\pi} \sum_{-N}^{N} e^{int}.$$

Collecting the even and odd terms, recall that

$$n \in \mathbb{N} \implies e^{int} + e^{-int} = 2\cos(nt), n > 0.$$

Hence, we can pair up all the terms ± 1, ± 2, etc, and write

$$D_N(t) = \frac{1}{2\pi} + \sum_{n=1}^{N} \frac{1}{\pi} \cos(nt).$$

So, we see that $D_N(t)$ is an even function. Moreover, we use the above expression to compute that

$$\int_{-\pi}^{\pi} D_N(t) dt = 1.$$

Since $D_N(t)$ is even, we also have:

dnint

$$\int_{-\pi}^{0} D_N(t) dt = \frac{1}{2} = \int_{0}^{\pi} D_N(t) dt. \tag{3.1}$$

The second observation is that $D_N(t)$ looks almost like a geometric series, but the problem is that it goes from minus exponents to positive ones. We can fix that right up by factoring out the largest negative exponent, so

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \sum_{n=0}^{2N} e^{int}.$$

We know how to sum a partial geometric series, don't we? This gives

dngeo

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \frac{1 - e^{i(2N+1)t}}{1 - e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})}. \tag{3.2}$$

To be continued. . .