FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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I hope you remember where we left off last time... We shall continue from that point, perhaps with a brief re-cap!

1.1. **Part 2 of the proof of Theorem on Pointwise Convergence of Fourier Series.** Now we return to our problem, in which we have

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt.$$

We want to show that $S_N(x)$ converges to the average of the right and left hand limits of f. In other words, this is equivalent to showing that

$$\lim_{N \to \infty} \left| S_N(x) - \frac{1}{2} \left(f(x_-) + f(x_+) \right) \right| = 0.$$

The S_N business has an integral, but the $f(x_\pm)$ don't. They have got a convenient factor of one half, so... Idea 5: Use our calculation of the integral of D_N to write

$$\frac{1}{2} f(x_-) = \int_{-\pi}^{0} D_N(t) dt f(x_-), \quad \frac{1}{2} f(x_+) = \frac{1}{2} = \int_{0}^{\pi} D_N(t) dt f(x_+).$$

Hence we are bound to prove that

$$\lim_{N \to \infty} \left| S_N(x) - \int_{-\pi}^{0} D_N(t) f(x_-) dt - \int_{0}^{\pi} D_N(t) f(x_+) dt \right| = 0.$$

Now, we use that

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt.$$

Hence, we want to show that

$$\left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \int_{-\pi}^{0} D_N(t) f(x_-) dt - \int_{0}^{\pi} D_N(t) f(x_+) dt \right| \to 0, \text{ as } N \to \infty.$$

It is quite natural to split things into two parts

$$\left| \int_{-\pi}^{0} D_N(t) (f(t+x) - f(x_-)) dt + \int_{0}^{\pi} D_N(t) (f(t+x) - f(x_+)) dt \right|.$$

Now, we know we've got to use the second expression for $D_N(t)$, and here's where it will come in handy. Let's insert it

$$\left| \int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x) - f(x_-))dt + \int_{0}^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x) - f(x_+))dt \right|.$$

Now, we know that if we take a function which is bounded, then its Fourier coefficients tend to 0, meaning $c_n \to 0$ as $|n| \to \infty$. We've got those e^{-iNt} and $e^{i(N+1)t}$ which look a lot like part of the definition of Fourier coefficient c_n for $|n|$ large... However, we've got this integrand defined two different ways on either side of zero.

Idea 6: Define a new function

$$g(t) = \frac{f(t+x) - f(x_-)}{1-e^{it}}, \qquad \text{for } t < 0,$$

$$g(t) = \frac{f(t+x) - f(x_+)}{1-e^{it}}, \qquad \text{for } t > 0.$$

How to define this function at zero? Let's look at the limit

$$\lim_{t\to 0_-} \frac{f(t+x) - f(x_-)}{1-e^{it}} = \lim_{t\to 0_-} \frac{t(f(t+x) - f(x_-))}{t(1-e^{it})} = \frac{f'(x_-)}{-ie^{i0}} = if'(x_-).$$

For the other side, a similar argument shows that

$$\lim_{t\to 0_+} \frac{f(t+x) - f(x_-)}{1-e^{it}} = if'(x_+).$$

So, depending upon whether $f'(x_-) = f'(x_+)$ or not, the function g will be continuous at 0, or not. However, even if it's not continuous, it is at least piecewise continuous, as well as piecewise differentiable, just like the original function f is. To see this, we see that for all other points $t \in [-\pi, \pi]$, the denominator of g is non-zero, and the numerator has the same properties as f. Therefore the above shows that g is indeed quite a lovely function on $[-\pi, \pi]$. The most important fact is that it is bounded on the closed interval $[-\pi, \pi]$, and hence its Fourier coefficients tend to zero by Bessel's inequality. This follows from the fact that any bounded function on a bounded interval, like $[-\pi, \pi]$, is in L^2 on that interval, i.e. in $L^2([-\pi, \pi])$.

Hence, we are looking at

$$\lim_{N\to\infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-iNt}dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(-N-1)t} g(t)dt \right| = \lim_{N\to\infty} |c_N(g) - c_{-N-1}(g)|,$$

where above, $c_N(g)$ is the N^{th} Fourier coefficient of g,

$$c_N(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-iNt}dt,$$

and similarly, $c_{-N-1}(g)$ is the $-N-1^{st}$ Fourier coefficient of g,

$$c_{-N-1}(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(-N-1)t} g(t)dt.$$

Idea 7: Use Bessel's inequality to say that

$$c_N(g) \to 0 \text{ as } N \to \infty, \qquad \text{and } c_{-N-1}(g) \to 0 \text{ as } N \to \infty.$$

Hence

$$\lim_{N\to\infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-iNt}dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(-N-1)t} g(t)dt \right| = |0+0| = 0.$$

Corollary 1. *Assume that f and g are piecewise $\mathcal{C}^1$, 2π periodic and have the same Fourier coefficients. Assume that at all points of discontinuity we have*

$$f(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right),$$

$$g(x) = \frac{1}{2}\left(g(x_-) + g(x_+)\right).$$

Then $f(x) = g(x)$ holds for all $x \in \mathbb{R}$.

Proof: By the theorem, for all $x \in \mathbb{R}$, and by the assumptions of the corollary, f and g have the same Fourier series, hence the same partial sums $S_N(x)$ as defined in the proof of the theorem. Therefore, by the theorem,

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right) = \frac{1}{2}\left(g(x_-) + g(x_+)\right).$$

The stuff with f is equal to $f(x)$ when f is continuous at x and also when f is not continuous at x. The stuff with g is similarly equal to $g(x)$ when g is continuous at x as well as when g is not continuous at x. Hence for all points $x \in \mathbb{R}$ we get $f(x) = g(x)$.

1.2. **Vibrating string example.** Let's do a PDE application - a vibrating string! Assume that at $t = 0$, the ends of the string are fixed, and we have pulled up the middle of it. This makes a shape which mathematically is described by the function

$$v(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

We assume that at $t = 0$ the string is not yet vibrating, so the initial conditions are then

$$\begin{cases} u(x,0) = v(x) \\ u_t(x,0) = 0 \end{cases}$$

We assume the ends of the string are fixed, so we have the boundary conditions

$$u(0) = u(2\pi) = 0.$$

Well, I suppose we should note that the string is identified with the interval $[0, 2\pi]$. We use our first technique, separation of variables. Remember, this is just the means to an end!! We have the wave equation

$$\Box u = 0, \quad \Box u = \partial_{tt} u - \partial_{xx} u.$$

Write

$$u(x,t) = X(x)T(t).$$

I used f and g before, but it seems that you students really like the BIG X and the BIG T. Okay, fine with me. So, we put them into the wave equation, assuming *for now* that $u(x,t) = X(x)T(t)$[1]

$$X(x)T''(t) - X''(x)T(t) = 0.$$

[1] In the end, $u(x,t)$ is *not* going to be of this form! So, this technique is just a means to an end.

We again *separate the variables* by dividing the whole equation by $X(x)T(t)$. Then we have

$$\frac{T''(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \text{ constant.}$$

The two sides depend on different variables, which makes them both have to be constant. We give that a name, λ. Then, since we have those handy dandy boundary conditions for X (but a much more complicated initial condition for $u(x,0) = v(x)$) we start with X. We solve

$$X'' = \lambda X, \quad X(0) = X(2\pi) = 0.$$

The cases $\lambda \geq 0$ won't satisfy the boundary condition. I leave it as an exercise for you to compute this. Do it! We are left with $\lambda < 0$ which by our old multivariable calculus theorem tells us that

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To get $X(0) = 0$, we must have $a = 0$. To get $X(2\pi) = 0$ we will need

$$\sqrt{|\lambda|}2\pi = k\pi \quad k \in \mathbb{Z}.$$

Hence

$$\sqrt{|\lambda|} = \frac{k}{2}, \quad k \in \mathbb{Z}.$$

Actually, because $\sin(-x) = -\sin(x)$ are linearly dependent, we only need to take $k \in \mathbb{N}$ (without 0, you know, American $\mathbb{N}$). So, we have X which we index by n, writing

$$X_n(x) = \sin(nx/2) \quad n \in \mathbb{N}.$$

For now, we don't worry about the constant factor. Next, we have the equation for the partner-function (can't forget the partner function!)

$$\frac{T_n''}{T_n} = \lambda_n.$$

Since we know that $\lambda_n < 0$ and $\sqrt{|\lambda_n|} = n/2$ we have

$$\lambda_n = -\frac{n^2}{4}.$$

Hence, our handy dandy multivariable calculus theorem tells us that the solution

$$T_n(t) = a_n\cos(nt/2) + b_n\sin(nt/2).$$

Now, we have

$$u_n(x,t) = X_n(x)T_n(t), \quad \Box u_n = 0 \quad \forall n \in \mathbb{N}.$$

Hence, we also have

$$\Box \sum_{n\geq 1} u_n(x,t) = \sum_{n\geq 1} \Box u_n(x,t) = 0,$$

because $\Box$ is a linear partial differential operator. We don't know which of these u_n guys we need to build our solution according to the initial conditions, so we just take all of them for now and chuck them out later if we don't need them.

So, we now need

$$u(x,t) := \sum_{n\geq 1} u_n(x,t)$$

to satisfy the initial condition. The easiest of these is the one that has zero on the right, namely $u_t(x,0) = 0$. So, we differentiate $u(x,t)$ with respect to t and set $t = 0$,

$$u_t(x,t) = \sum_{n\geq 1} X_n(x) T_n'(0) = \sum_{n\geq 1} X_n(x) \left(-a_n \frac{n}{2}\sin(0) + b_n \frac{n}{2}\cos(0)\right)$$

$$= \sum_{n\geq 1} X_n(x) b_n.$$

We need this to be the zero function. Basically, we are expanding the zero function in terms of the basis functions X_n. In the theorem for pointwise convergence of Fourier series, we used the c_n's. This was for convenience. The same theorem holds when we use the sine and cosine expansion like here. They're all equivalent. Of course, if we think about how to get the expansion in terms of the basis, the coefficients will be the scalar product of the zero function and X_n. This will be 0. So, the coefficients $b_n = 0$ for all n.

Now we use the other initial condition to get the coefficients a_n,

$$u(x,0) = \sum_{n\geq 1} X_n(x) a_n.$$

We want this to be equal to

$$v(x).$$

Although we've been working so far with the interval $[-\pi, \pi]$, this is basically the same. We are now on the interval $[0, 2\pi]$. The coefficients will be

$$\frac{1}{||X_n||^2}\langle v, X_n\rangle = \frac{1}{\pi}\int_0^{2\pi} X_n(x) v(x) dx.$$

What happened to the complex conjugation? Well, it is there, it just ain't doing nothin to v because v is real valued. Also, I leave it as an exercise to compute that

$$||X_n||^2 = \int_0^{2\pi} \sin(nx/2)^2 dx = \pi.$$

So, now is just to compute

$$\int_0^{\pi} \sin(nx/2) x dx + \int_{\pi}^{2\pi} \sin(nx/2)(2\pi - x) dx.$$

I leave this also as an exercise (you can use BETA :-)

1.3. **Derivatives and Fourier series.** We'll prove a few facts about how the derivatives of functions interact with the Fourier series. The next one is a theory item (it can appear on the exam). So stay sharp!

Theorem 2. *Assume that f is 2π periodic, continuous, and piecewise $\mathcal{C}^1$. Then f' has Fourier coefficients a_n', b_n', c_n' with*

$$a_n' = nb_n, \quad b_n' = -na_n, \quad c_n' = inc_n.$$

Proof: DO NOT DIFFERENTIATE THE SERIES TERMWISE!!!!!!! THE ARGUMENT WILL BE CIRCULAR AND TOTALLY WRONG!!!!!

Instead, use the definition and integrate by parts. By now, we can give the equivalent formulation of

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx.$$

You can check using the linearity of the integral that this is what you get by definition of c_n together with the relationship between c_n and a_n. Anyhow, take this and do integration by parts.

$$a_n = \frac{1}{\pi} \left. \frac{\sin(nx)}{n} f(x) \right|_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \frac{\sin(nx)}{n} dx.$$

The first term with sine vanishes, and the second term is by definition

$$-\frac{b_n'}{n}.$$

Hence, re-arranging, we see that

$$b_n' = -n a_n.$$

Similarly, we use integration by parts for the

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \left. \frac{-\cos(nx)}{n} f(x) \right|_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \frac{\cos(nx)}{n} dx.$$

The first term evaluates to zero be periodicity. So, we get the second term which is by definition

$$\frac{a_n}{n}.$$

Hence

$$n b_n = a_n.$$

Finally, we integrate by parts for the c_n,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \left. \frac{f(x) e^{-inx}}{-in} \right|_{-\pi}^{\pi} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f'(x) e^{inx}}{in} dx = \frac{c_n'}{in}.$$

Thus, re-arranging

$$in c_n = c_n'.$$

We can use this to prove

Theorem 3. *Assume that f is 2π periodic, continuous, and piecewise $\mathcal{C}^1$. Then the Fourier series of f converges absolutely uniformly to f on all of $\mathbb{R}$!*

Proof: By assumption, f' is piecewise continuous. Bessel's equation tells us that

$$\sum_{\mathbb{Z}} |c_n'|^2 < \infty.$$

We use the preceding theorem to say that for all $n \neq 0$,

$$|c_n| = \left| c_n' \frac{1}{n} \right|.$$

Hence we can estimate

$$\sum_{n\in\mathbb{Z}} |c_n e^{inx}| = \sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c'_n|}{|n|}.$$

There is also a Cauchy-Schwarz theorem for what is known as "little l^2". Little l^2 is the set of all sequences $\{z_n\}_{n\in\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} |z_n|^2 < \infty.$$

The norm on little l^2 is

$$||\{z_n\}|| = \sqrt{\sum_{n\in\mathbb{Z}} |z_n|^2}.$$

Hence, by Bessel's inequality

$$\sum_{n\in\mathbb{Z}} |c'_n|^2 < \infty,$$

and we know very well that

$$\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2} < \infty,$$

we see that

$$\sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c'_n|}{|n|} \le |c_0| + \sqrt{\sum_{n\in\mathbb{Z}} |c'_n|^2} + \sqrt{\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2}} < \infty.$$

Therefore the Fourier series converges absolutely, and uniformly for all $x \in \mathbb{R}$, because we see that the convergence estimates are independent of the point x. Since the function is continuous, the limit of the series is always $f(x)$.

Let's do an example of how we can use Fourier series to sum infinite series!

1.4. **Example of using Fourier series to compute sums.** We shall compute:

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2}.$$

Hint: Expand e^x in a Fourier series on $(-\pi,\pi)$). Often, you'll be given such a hint, as when this problem appeared on an exam...

Okay, we follow the hint. We need to compute

$$\int_{-\pi}^{\pi} e^x e^{-inx} dx = \left.\frac{e^{x(1-in)}}{1-in}\right|_{x=-\pi}^{x=\pi} = \frac{e^{\pi}e^{-in\pi}}{1-in} - \frac{e^{-\pi}e^{in\pi}}{1-in} = (-1)^n \frac{2\sinh(\pi)}{1-in}.$$

Hence, the Fourier coefficients are

$$\frac{1}{2\pi}(-1)^n \frac{2\sinh(\pi)}{1-in},$$

and the Fourier series for e^x on this interval is

$$e^x = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)} e^{inx}, \quad x \in (-\pi,\pi).$$

We can pull out some constant stuff,

$$e^x = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{1-in}, \quad x \in (-\pi, \pi).$$

Now, we use the theorem which tells us that the series converges to the average of the left and right hand limits at points of discontinuity, like for example π. The left limit is e^π. Extending the function to be 2π periodic, means that the right limit approaching π is equal to $e^{-\pi}$. Hence

$$\frac{e^\pi + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\pi}}{1-in}.$$

Now, we know that $e^{in\pi} = (-1)^n$, thus

$$\frac{e^\pi + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{1}{1-in}.$$

We now consider the sum, and we pair together $\pm n$ for $n \in \mathbb{N}$, writing

$$\sum_{-\infty}^{\infty} \frac{1}{1-in} = 1 + \sum_{n \in \mathbb{N}} \frac{1}{1-in} + \frac{1}{1+in} = 1 + \sum_{n \in \mathbb{N}} \frac{2}{1+n^2}.$$

Hence we have found that

$$\frac{e^\pi + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\pi}}{1-in} = \frac{\sinh(\pi)}{\pi} \left(1 + \sum_{n \in \mathbb{N}} \frac{2}{1+n^2} \right).$$

The rest is mere algebra. On the left we have the definition of $\cosh(\pi)$. So, moving over the $\sinh(\pi)$ we have

$$\frac{\pi \cosh(\pi)}{\sinh(\pi)} = 1 + 2 \sum_{n \in \mathbb{N}} \frac{1}{1+n^2} \implies \left(\frac{\pi \cosh(\pi)}{\sinh(\pi)} - 1 \right) \frac{1}{2} = \sum_{n \in \mathbb{N}} \frac{1}{1+n^2}.$$

Wow.