FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Today we will continue to look at Fourier series as well as consequences for derivatives and integrals of functions which have Fourier series. To make things precise, we need a definition. However, lucky for you, this definition is *not* one that you need to memorize.

Definition 1. Let

$$\ell^2(\mathbb{C}) := \{(z_n)_{n\in\mathbb{Z}}, \quad z_n \in \mathbb{C}\forall n, \text{ and } \sum_{n\in Z} |z_n|^2 < \infty\}.$$

This is a Hilbert space (complete normed vector space with scalar product) with the scalar product

$$\langle z, w\rangle := \sum_{n\in\mathbb{Z}} z_n \overline{w_n}, \quad z = (z_n)_{n\in Z}, \quad w = (w_n)_{n\in\mathbb{Z}}.$$

The norm on the Hilbert space, $\ell^2 = \ell^2(\mathbb{C})$ is defined by

$$||z|| = \sqrt{\sum_{n\in\mathbb{Z}} |z_n|^2}.$$

We also have a Cauchy-Schwarz inequality:

$$|\langle z, w\rangle| \le ||z||||w||.$$

We will use this together with the relationship between the Fourier coefficients for a piecewise $\mathcal{C}^1$ and continuous function, f, to prove

Theorem 2. *Assume that f is 2π periodic, continuous, and piecewise $\mathcal{C}^1$. Then the Fourier series of f converges absolutely uniformly to f on all of $\mathbb{R}$!*

Proof: By assumption, f' is piecewise continuous. Bessel's equation tells us that

$$\sum_{\mathbb{Z}} |c_n'|^2 < \infty.$$

We use the preceding theorem to say that for all $n \neq 0$,

$$|c_n| = \left|c_n' \frac{1}{n}\right|.$$

Hence we can estimate

$$\sum_{n\in\mathbb{Z}} |c_n e^{inx}| = \sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c_n'|}{|n|}.$$

By Bessel's inequality

$$\sum_{n\in\mathbb{Z}} |c_n'|^2 < \infty,$$

and we know very well that

$$\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2} < \infty.$$

So, using the Cauchy-Schwarz inequality on ℓ^2, we have

$$\sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c_n'|}{|n|} \le |c_0| \sqrt{\sum_{n\in\mathbb{Z}} |c_n'|^2} + \sqrt{\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2}} < \infty.$$

Therefore the Fourier series converges absolutely, and uniformly for all $x \in \mathbb{R}$, because we see that the convergence estimates are independent of the point x. Since the function is continuous, the limit of the series is always $f(x)$.

We can repeat this idea to show that the more derivatives a function has, the faster its Fourier series converges.

Theorem 3. *Let f be 2π periodic, and assume that f is $\mathcal{C}^{k-1}$, and $f^{(k-1)}$ is piecewise $\mathcal{C}^1$, and f is piecewise $\mathcal{C}^k$. Then the Fourier coefficients of f satisfy*

$$\sum |n^k a_n|^2 < \infty, \quad \sum |n^k b_n|^2 < \infty, \quad \sum |n^k c_n|^2 < \infty.$$

If $|c_n| \le c|n|^{-k-\alpha}$ for some $c > 0$ and $\alpha > 1$, for all $n \neq 0$, then $f \in \mathcal{C}^k$.

Proof: We apply the theorem relating the Fourier coefficients of f to those of the derivatives of f. Do it k times. We get

$$c_n^{(k)} = (in)^k c_n.$$

Next, we apply Bessel's inequality to conclude that since f is piecewise $\mathcal{C}^k$, $f^{(k)}$ is bounded on the interval hence it is in L^2 on the interval, and so

$$\sum |c_n^{(k)}|^2 < \infty.$$

Since

$$|c_n^{(k)}| = |n|^k |c_n|$$

this shows that

$$\sum |n^k c_n|^2 < \infty.$$

We have similar estimates for a_n and b_n using the same theorem, specifically

$$|a_n^{(k)}| = |n^k a_n|, \quad |b_n^{(k)}| = |n^k b_n|.$$

Hence,

$$\sum |n^k a_n| < \infty, \quad \sum |n^k b_n| < \infty.$$

For the last statement, since we assumed that f is $\mathcal{C}^{k-1}$, we let

$$g(x) := f^{(k-1)}(x).$$

Then g is continuous and by assumption it is piecewise $\mathcal{C}^1$. Therefore, by the big convergence theorem, the Fourier series of g converges to (really, it is equal to!) $g(x)$ for all x in $\mathbb{R}$. Next, we use the assumption and the fact that the Fourier coefficients of g are

$$c_n^{(k-1)} = (in)^{k-1} c_n.$$

Therefore

$$\sum_{n \in \mathbb{Z}} |c_n^{(k-1)} e^{inx}| = \left| c_0^{(k-1)} \right| + \sum_{n \neq 0} |n^{k-1}||c_n| \leq \left| c_0^{(k-1)} \right| + c \sum_{n \neq 0} |n|^{k-1-k-\alpha} < \infty.$$

Hence, the series converges absolutely and uniformly in $\mathbb{R}$. Moreover, differentiating the series termwise is legitimate, because the result

$$\sum_{n \in \mathbb{Z}} in c_n^{(k-1)} e^{inx}$$

also converges absolutely and uniformly in $\mathbb{R}$:

$$\sum_{n \in \mathbb{Z}} |in c_n^{(k-1)}| \leq \sum_{n \neq 0} |n||c_n^{(k-1)}| \leq c \sum_{n \neq 0} |n||n|^{k-1-k-\alpha} < \infty$$

because $\alpha > 1$. Since the series is equal to $g(x) = f^{(k-1)}(x)$ for all $x \in \mathbb{R}$, and the series is a differentiable function for all $x \in \mathbb{R}$, this shows that g is differentiable for all $x \in \mathbb{R}$. Moreover, g' is continuous on $\mathbb{R}$, because the series defines a continuous function.[1] This is the case because the series defining g' converges absolutely and uniformly for all of $\mathbb{R}$. Hence, $f^{(k-1)}$ is in $\mathcal{C}^1$ on all of $\mathbb{R}$, and therefore f is in $\mathcal{C}^k$ on all of $\mathbb{R}$.

We now prove a theorem about integrating Fourier series.

Theorem 4. *Let f be a 2π periodic function which is piecewise continuous. Define*

$$F(x) := \int_0^x f(t)dt.$$

If $c_0 = 0$, then

$$F(x) = C_0 + \sum_{n \neq 0} \frac{c_n}{in} e^{inx}, \quad C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)dx.$$

Similarly,

$$F(x) = \frac{1}{2} A_0 + \sum_{n \geq 1} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \cos(nx).$$

Proof: We first note that F is continuous and piecewise $\mathcal{C}^1$, because it is the integral of a piecewise continuous function. Moreover, assuming $c_0 = 0$, we see that

$$F(x+2\pi) - F(x) = \int_0^{x+2\pi} f(t)dt - \int_0^x f(t)dt = \int_x^{x+2\pi} f(t)dt = \int_{-\pi}^{\pi} f(t)dt = 2\pi c_0 = 0.$$

Above we have used the nifty lemma that allows us to slide around integrals of periodic functions. So, F satisfies the assumptions of the theorem on pointwise convergence of Fourier series. We therefore have pointwise convergence of the Fourier

[1] This is true because the series should really be viewed as the limit of the partial series, and each partial series defines a smooth, thus also continuous, function. The uniform limit of continuous functions is itself a continuous function.

series of F. Moreover, applying the theorem relating the Fourier coefficients of $F' = f$ to those of F, we have

$$C_n = \frac{c_n}{in} \quad n \neq 0.$$

(That's because $c_n = C_n'$ and the theorem says $C_n' = inC_n$ which shows $c_n = inC_n$, which we can re-arrange as above). Of course, the formula for C_0 is just the usual formula for it, because we can't say anything more specific without knowing more information on f. The re-statement in terms of a and b follows from the relationship between these and the c_n.

Remark 1. If $c_0 \neq 0$, then define a new function

$$g(t) := f(t) - c_0.$$

Since f is 2π periodic, so is g. Then, apply the theorem above to g. Note that

$$G(x) = \int_0^x g(t)dt = F(x) - c_0 x.$$

Moreover, the Fourier coefficients of g,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - c_0)e^{-inx}dx = c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad \forall n \neq 0.$$

So, the series for $G(x)$ from the theorem is

$$\widetilde{C_0} + \sum_{n\neq 0} \frac{c_n}{in} e^{inx},$$

with

$$\widetilde{C_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(F(x) - c_0 x\right) dx = C_0.$$

So, in fact, it is the same C_0, where we have used the oddness of the function x above. Then, we get something of a corollary which says that in general, the series in the theorem,

$$C_0 + \sum_{n\neq 0} \frac{c_n}{in} e^{inx}, \quad C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx$$

converges to $F(x) - c_0 x$.

1.1. **Using Fourier series to compute sums.** Let's figure out how to use a Fourier series to compute

$$\sum_{n\geq 1} \frac{1}{n^4}.$$

For starters, we expand x^2 in a Fourier series. This is an even function, hence no sines in its Fourier series. The other terms

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)dx.$$

We do this integral:

$$\int_0^{\pi} x^2 \cos(nx)dx = \int x^2 \left(\frac{\sin(nx)}{n}\right)' dx = x^2 \frac{\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x \frac{\sin(nx)}{n} dx.$$

Above we did integration by parts. The first part vanishes. The second term we handle with integration by parts again,

$$\int_0^\pi x\sin(nx)dx = \int_0^\pi x\left(-\cos(nx)/n\right)' dx = -\frac{x\cos(nx)}{n}\bigg|_0^\pi + \int_0^\pi \cos(nx)/ndx.$$

Now this time the second term vanishes because integrating gives us a sine which is 0 at 0 and at π. So, recalling the constant factors, we get

$$\int_0^\pi x^2\cos(nx)dx = \frac{2\pi\cos(\pi n)}{n^2} = \frac{2\pi(-1)^n}{n^2}.$$

Hence our coefficients,

$$a_n = \frac{2 * 2(-1)^n}{n^2}.$$

Moreover, we also compute the term

$$a_0 = \frac{1}{\pi}\int_{-\pi}^\pi x^2 dx = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}.$$

Hence, the Fourier series expansion of x^2 is

$$\frac{\pi^2}{3} + 4\sum_{n\geq 1}\frac{(-1)^n\cos(nx)}{n^2}.$$

Let $x = \pi$. Since our periodically extended function, x^2 is continuous on all of $\mathbb{R}$, the Fourier series converges to its value at $x = \pi$ which means

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n\geq 1}\frac{(-1)^n(-1)^n}{n^2} \implies \frac{\pi^2}{6} = \sum_{n\geq 1}\frac{1}{n^2}.$$

To get up to summing n^{-4} we use Theorem 2.4 about integrating Fourier series. We see that

$$c_0 = \frac{\pi^2}{3}.$$

We also see that for $f(t) = t^2$,

$$F(x) := \int_0^x f(t)dt = \frac{x^3}{3}.$$

The series from the theorem is

$$C_0 + 4\sum_{n\geq 1}\frac{(-1)^n\sin(nx)}{n^3}.$$

The term

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^\pi F(x)dx = 0,$$

because $F(x)$ above is odd. Hence, the theorem together with the remark after it says that

$$4\sum_{n\geq 1}\frac{(-1)^n\sin(nx)}{n^3} = \frac{x^3}{3} - \frac{\pi^2 x}{3}, \quad x\in[-\pi,\pi].$$

Exercise: Compute $\sum n^{-3}$.

To proceed, we're going to need to use the theorem once more to get n^4 in the denominator. Before we do this, let's multiply everything by 3 to make it nicer. Then

$$x^3 - \pi^2 x = 12 \sum_{n \geq 1} \frac{(-1)^n \sin(nx)}{n^3}, \quad x \in [-\pi, \pi].$$

So, here we have

$$f(t) = t^3 - \pi^2 t \implies F(x) = \int_0^x f(t) dt = \frac{x^4}{4} - \frac{\pi^2 x^2}{2}.$$

We see also that

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt = 0.$$

Hence, we apply the theorem directly to F. The theorem says

$$F(x) = C_0 + 12 \sum_{n \geq 1} -\frac{(-1)^n \cos(nx)}{n^4}.$$

We compute

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx = \frac{1}{\pi} \int_0^{\pi} \frac{x^4}{4} - \frac{\pi^2 x^2}{2} dx = \frac{\pi^4}{20} - \frac{\pi^4}{6}.$$

Therefore

$$F(x) = \frac{x^4}{4} - \frac{\pi^2 x^2}{2} = \frac{\pi^4}{20} - \frac{\pi^4}{6} - 12 \sum_{n \geq 1} \frac{(-1)^n \cos(nx)}{n^4}, \quad x \in [-\pi, \pi].$$

We do the same trick now of choosing

$$x = \pi \implies \cos(nx) = \cos(n\pi) = (-1)^n, \quad (-1)^n (-1)^n = 1 \forall n.$$

Hence,

$$F(\pi) = \frac{\pi^4}{4} - \frac{\pi^4}{2} = \frac{\pi^4}{20} - \frac{\pi^4}{6} - 12 \sum_{n \geq 1} \frac{1}{n^4}.$$

Re-arranging things

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{1}{12} \left(\frac{\pi^4}{20} - \frac{\pi^4}{6} + \frac{\pi^4}{2} - \frac{\pi^4}{4} \right).$$

Just for fun, we determine what this is...

$$\frac{\pi^4}{20} - \frac{\pi^4}{6} + \frac{\pi^4}{2} - \frac{\pi^4}{4} = \frac{\pi^4}{2} \left(\frac{1}{10} - \frac{1}{3} + \frac{1}{2} \right) = \frac{\pi^4}{2} \left(\frac{3 - 10 + 15}{30} \right)$$

$$= \frac{\pi^4}{2} \left(\frac{8}{30} \right) = \frac{2\pi^4}{15}.$$

So, recalling the factor of $\frac{1}{12}$, we see that

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{2\pi^4}{(12)(15)} = \frac{\pi^4}{6(15)} = \frac{\pi^4}{90}.$$

Wow, who would have guessed that? Not I said the fly!

1.2. **Fourier sine and cosine series.** Let's say we are just looking at $[0, \pi]$. There are two ways to extend a function defined over there to all of $[-\pi, \pi]$. One way is oddly, and the other way is evenly. If we want to extend oddly, we define

$$f(x) := -f(-x), \quad x \in (-\pi, 0).$$

This could create a discontinuity at $x = 0$, but no worry. Then, we have computed in an exercise that the a_n coefficients are all zero, and the b_n coefficients are

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx.$$

Here we used the fact that sine is also an oddball. On the other hand, if we want to extend evenly, we define

$$f(x) := f(-x), \quad x \in (-\pi, 0).$$

Then, we have computed in an exercise that the b_n are all zero, because our function is even. Here we have the coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx.$$

Above we used the fact that cosine is even. In this way, we may define Fourier sine and cosine series for functions on $[0, \pi]$. The Fourier sine series is defined to be

$$\sum_{n \geq 1} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

whereas the Fourier cosine series is

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(nx), \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx, \quad \forall n \in \mathbb{N}.$$

Theorem 5. *Let f be a function which is piecewise $\mathcal{C}^1$ on $[0, \pi]$. Then the Fourier sine and cosine series converge to $f(x)$ for all $x \in (0, \pi)$ at which f is continuous. For other points, they converge to*

$$\frac{1}{2} \left(f(x_-) + f(x_+) \right).$$

Proof: First, we extend the function either evenly or oddly. Next, we extend it to all of $\mathbb{R}$ to be 2π periodic. Like Riker, we just *make it so.* We're only proving a statement about points in $(0, \pi)$. So, what happens outside of this set of points, well it don't matter. We apply the theorem on pointwise convergence of Fourier series now.