# FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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We don't need to start with a periodic function. We can just take any old function on any old interval and use our methods. Here's how to do that. For a function $f$ defined on an interval $[a-\ell, a+\ell]$ for some $a \in \mathbb{R}$, and some $\ell > 0$, we begin by extending $f$ to be $2\ell$ periodic on $\mathbb{R}$. Next, we define

$$g(t) := f\left(\frac{t\ell}{\pi} + a\right) = f(x),$$

that is

$$\frac{t\ell}{\pi} + a = x, \quad t = \frac{(x-a)\pi}{\ell}.$$

Then, the function $g(t)$ is $2\pi$ periodic, because

$$g(t+2\pi) = f\left(\frac{(t+2\pi)\ell}{\pi} + a\right) = f\left(\frac{t\ell}{\pi} + a + 2\ell\right) = f\left(\frac{t\ell}{\pi} + a\right).$$

Above, we used the fact that $f$ is $2\ell$ periodic. So, now that we got $g$, we just do all our Fourier series magic to $g$. Presuming $g$ is not too terrible, we can expand $g$ in a Fourier series,

$$g(t) = \sum_{n \in \mathbb{Z}} c_n e^{int}.$$

Then, we get by substituting for $t$ in terms of $x$

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{in\left(\frac{(x-a)\pi}{\ell}\right)}.$$

Here we note that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{t\ell}{\pi} + a\right)e^{-int}dt.$$

Substituting in the integral,

$$x = \frac{t\ell}{\pi} + a, \quad dx = \frac{\ell dt}{\pi}$$

$$c_n = \frac{1}{2\pi}\frac{\pi}{\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx = \frac{1}{2\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx.$$

So, we can work our Fourier-series magic on basically any arbitrary function we like!

1.1. **Functions as infinite dimensional vectors.** Let us fix an interval, for now we fix the interval $[-\pi, \pi]$, but by the observations above, we can do the same on any arbitrary *finite* interval. We have a function defined on the interval. Let's call it $f$. Assume that

fl2
$$\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty. \tag{1.1}$$

Then, by the Cauchy-Schwarz inequality

$$|c_n| := \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right| \leq \frac{1}{2\pi} \sqrt{\int_{-\pi}^{\pi} |f(x)|^2 dx} \sqrt{\int_{-\pi}^{\pi} 1 dx} = \frac{1}{2\pi} ||f|| \sqrt{2\pi} = \frac{||f||}{2\pi}.$$

So, the $c_n \in \mathbb{C}$ are all finite. Do you remember what $||f||$ is? Next, we recall Bessel's inequality:

$$\sum_{n \in \mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{||f||^2}{2\pi}.$$

So, if you forgot what $||f||$ is, you can figure it out from above. In particular, since we started out by assuming $f$ satisfies (fl2 1.1), we see that the sequence

$$(c_n)_{n \in \mathbb{Z}} \in \ell^2.$$

So, for any $f \in \mathcal{L}^2$ on the interval $[-\pi, \pi]$, we can associate a sequence in $\ell^2$. As we get into the general theory of Hilbert spaces, it's going to be useful to introduce the notation

$$c_n = \hat{f}_n = \hat{f}(n).$$

Now, let's assume that for some $f$ in $\mathcal{L}^2$ and some $g$, they have all the same Fourier coefficients. What I am about to explain is extremely subtle. It may require some really careful thinking, or if it bothers you, just forget about it.

1.1.1. $\mathcal{L}^2$ *convergence versus pointwise convergence.* For $f$ as above, defined on $[-\pi, \pi]$, let's define

$$F(x) := \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Note that we have *only* assumed (fl2 1.1). No continuity, no piecewise $\mathcal{C}^1$, none of that stuff. So, this "function" $F(x)$ might not be defined everywhere. What we're going to prove is that:

$$\int_{-\pi}^{\pi} |F(x) - f(x)|^2 dx = 0.$$

First, we compute

$$\int_{-\pi}^{\pi} |F(x)|^2 dx = \int F(x) \overline{F(x)} dx = \int \sum_{n \in \mathbb{Z}} c_n e^{inx} \overline{\sum_{m \in \mathbb{Z}} c_m e^{imx}}$$

$$= \sum_{m,n \in \mathbb{Z}} c_n \overline{c_m} \int e^{inx} \overline{e^{imx}} dx = \sum_{m,n \in \mathbb{Z}} c_n \overline{c_m} \langle e^{inx}, e^{imx} \rangle.$$

We computed way at the beginning that

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$$\langle e^{inx}, e^{imx}\rangle = \begin{cases} 0 & n \neq m \\ 2\pi & n = m. \end{cases} \tag{1.2}$$

Hence, the terms in the sum with $n \neq m$ are all zero. We only have $n = m$, so we can just write $n$ for both. Thus we get

$$\sum_{n\in\mathbb{Z}} |c_n|^2 2\pi = 2\pi \sum_{n\in\mathbb{Z}} |c_n|^2 \le ||f||^2 < \infty.$$

This means that $F(x)$ is *also in $\mathcal{L}^2$ on the interval.* It requires a little more machinery than what we have currently, but with the power of Hilbert spaces and the Stone-Weierstrass theorem, we will prove that:

$$\{e^{inx}\}_{n\in\mathbb{Z}} \text{ is an orthogonal basis for } \mathcal{L}^2 \text{ on } [-\pi, \pi].$$

As a consequence, we will prove that Bessel's inequality is actually an *equality* in this case. For now, please just accept this as a fact. A consequence is that

$$||F||^2 = 2\pi \sum_{n\in\mathbb{Z}} |c_n|^2 = ||f||^2.$$

So, $F$ and $f$ have the same $\mathcal{L}^2$ norm.

Next, we compute the $\mathcal{L}^2$ distance between $F$ and $f$:

$$\int_{-\pi}^{\pi} \left(\sum_{n\in\mathbb{Z}} c_n e^{inx} - f(x)\right) \overline{\left(\sum_{m\in\mathbb{Z}} c_m e^{imx} - f(x)\right)} dx$$

$$= \int (\sum_n c_n e^{inx} - f(x))(\sum_m \overline{c_m} e^{-imx} - \overline{f(x)}) dx$$

$$= \sum_{n,m\in\mathbb{Z}} \int \left(c_n e^{inx}\overline{c_m} e^{-imx} - c_n e^{inx}\overline{f(x)} - \overline{c_m} e^{-imx} f(x)\right) dx + \int |f(x)|^2 dx$$

$$= \sum_{n,m\in\mathbb{Z}} c_n\overline{c_m} \int e^{inx}e^{-imx}dx - \sum_{n\in\mathbb{Z}} c_n \int e^{inx}\overline{f(x)}dx - \sum_{m\in\mathbb{Z}} \overline{c_m} \int e^{-imx} f(x)dx + \int |f(x)|^2 dx.$$

Note that

$$\int e^{inx}\overline{f(x)} = \int \overline{e^{-inx} f(x)} = \overline{\int f(x) e^{-inx}} = 2\pi\overline{c_n}.$$

Hence, this together with the definition of Fourier coefficients shows that what we have above is

$$\sum_{n,m\in\mathbb{Z}} c_n\overline{c_m} \int e^{inx}e^{-imx}dx - \sum_{n\in\mathbb{Z}} 2\pi c_n\overline{c_n} - \sum_{m\in\mathbb{Z}} 2\pi\overline{c_m}c_m + \int |f(x)|^2 dx.$$

By the same calculation as before, (eortho 1.2), this is

$$\sum_{n\in\mathbb{Z}} 2\pi c_n\overline{c_n} - 2\sum_{n\in\mathbb{Z}} 2\pi c_n\overline{c_n} + \int |f(x)|^2 dx$$

$$= ||f||^2 - 2\pi \sum_{n\in\mathbb{Z}} |c_n|^2 = 0.$$

This shows that the $\mathcal{L}^2$ *distance* between $f$ and $F$ is zero. Just like with vectors, the distance on a Hilbert space is

$$||f - F|| = \sqrt{\int |f(x) - F(x)|^2}.$$

So, $f$ is the same as its Fourier series, as elements of $\mathcal{L}^2$. The elements of $\mathcal{L}^2$ are not actually functions, but rather, equivalence classes of functions. To understand this, let's think about the equivalence class of the zero function. This consists of all functions such that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 0.$$

This is the case precisely when

$$f(x) = 0 \text{ for almost every } x \in [-\pi, \pi].$$

What does almost every mean? It means that there is a set which has one-dimensional Lebesgue measure equal to zero, call it $\mathcal{N}$, and

$$f(x) = 0 \forall x \in [-\pi, \pi] \setminus \mathcal{N}.$$

What does it mean to have one-dimensional Lebesgue measure equal to zero? Well, it means that the set, $\mathcal{N}$ has *no length.* For example, a single point has no length. Two points together also have no length. Any countably infinite set of points also has no length. So, $\mathcal{N}$ is some such set which has no length.

So, similarly,

$$F(x) = f(x) \text{ for almost every } x \in [-\pi, \pi].$$

This shows that we can uniquely identify the elements of $\mathcal{L}^2$ on the interval with the elements of $\ell^2$. The identification is

$$f \leftrightarrow (c_n)_{n \in \mathbb{Z}}.$$

On the one hand, for $f \in \mathcal{L}^2$, it gives us a sequence (the sequence of its Fourier coefficients) which is in $\ell^2$. The sequence is unique, because if $g$ has all the same Fourier coefficients as $f$, then

$$g(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \text{ almost everywhere, and } f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \text{ almost everywhere,}$$

so

$$g(x) = f(x) \text{ almost everywhere,}$$

thus

$$g = f \text{ as elements of } \mathcal{L}^2.$$

So, each element of $\mathcal{L}^2$ has a unique sequence associated to it. On the other hand, let $(c_n)_{n \in \mathbb{Z}} \in \ell^2$. Then, we can use it to define $f \in \mathcal{L}^2$ via

$$f(x) := \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

By definition of $\ell^2$,

$$\sum_{n \in \mathbb{Z}} |c_n|^2 < \infty.$$

By our above calculations

$$||f|| = \int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty.$$

So, indeed $f \in \mathcal{L}^2$.[1]

In summary, we can legitimately identify $\mathcal{L}^2$ functions with infinite dimensional but finite length vectors. Pretty cool. So, now we shall proceed with the general theory of (possibly infinite dimensional) Hilbert spaces.

1.2. **Hilbert spaces.** A *Hilbert space* is a complete[2], normed vector space whose norm is defined by a scalar product. The definition of a vector space means that if $u$ and $v$ are elements in your Hilbert space, then for all complex numbers $a$ and $b$,

$$au + bv \text{ is in your Hilbert space.}$$

So, taking $a = b = 0$, there is always a 0 vector in your Hilbert space. The fact that it is normed means that every element of the Hilbert space has a *length,* which is equal to its norm. To define this, we describe the scalar product. For a Hilbert space $H$, the scalar product satisfies:

$$u, v \in H \implies \langle u, v\rangle \in \mathbb{C},$$

$$c \in \mathbb{C} \implies \langle cu, v\rangle = c\langle u, v\rangle,$$

$$u, v, w \in H \implies \langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle,$$

$$\langle u, v\rangle = \overline{\langle v, u\rangle},$$

$$\langle u, u\rangle \geq 0, \quad = 0 \iff u = 0.$$

Therefore, we can define the norm of a vector as

$$||u|| := \sqrt{\langle u, u\rangle}.$$

The norm of a vector is also equal to its distance from the 0 element of the Hilbert space. Similarly,

$$||u - v|| = \sqrt{\langle u - v, u - v\rangle}$$

is the distance between the elements $u$ and $v$ in your Hilbert space. We say that a set of elements

$$\{u_\alpha\} \subset H$$

is an orthonormal basis (ONB) for $H$ if for any $v \in H$ there exist complex numbers $(c_\alpha)$ such that

$$v = \sum c_\alpha u_\alpha, \quad \langle u_\alpha, u_\beta\rangle = \delta_{\alpha,\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta. \end{cases}$$

This is the Kronecker $\delta$. You may be wondering why we haven't written an index for $\alpha$. Well, that's because à priori, they could be uncountable.

**Theorem 1.** *A Hilbert space is separable if and only if it has either a finite ONB or a countable ONB.*

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[1] If you're the type of person to worry about measurability, *don't*. The function defined this way is indeed measurable. If you're skeptical, I leave it as an **Exercise** to prove it!

[2] Every Cauchy sequence converges. Do you remember what a Cauchy sequence is? If not, please look it up or ask!

There is a cute proof here:
`http://www.polishedproofs.com/relationship-between-a-countable-orthonormal-basis-and-a-cou`

We're only going to be working with Hilbert spaces which have either a finite ONB or a countable ONB. The dimension of a Hilbert space is the number of elements in an ONB. Any finite dimensional Hilbert space is in bijection with the standard one

$$\mathbb{C}^n, \quad u, v \in \mathbb{C}^n \implies \langle u, v \rangle = u \cdot \overline{v}.$$

Thus, writing

$$u = (u_1, \ldots, u_n), \qquad \text{with each component } u_k \in \mathbb{C}, \ k = 1, \ldots, n$$

and similarly for $v$,

$$\langle u, v \rangle = \sum_{k=1}^{n} u_k \overline{v_k}.$$

The bijection between any finite (n) dimensional Hilbert space and $\mathbb{C}^n$ comes from taking an ONB of the Hilbert space and mapping the elements of the ONB to the standard basis vectors of $\mathbb{C}^n$. Here are some useful basic results for Hilbert spaces.

**Proposition 2.** *Let $H$ be a Hilbert space. For any $u$ and $v$ in $H$,*

$$||u + v||^2 = ||u||^2 + 2\Re\langle u, v \rangle + ||v||^2.$$

**Proof:** Compute:

$$\begin{aligned} ||u + v||^2 = \langle u + v, u + v \rangle &= \langle u, u + v \rangle + \langle v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, v \rangle + \langle v, u \rangle \\ &= ||u||^2 + \langle u, v \rangle + ||v||^2 + \overline{\langle u, v \rangle}. \end{aligned}$$

We all know that for a complex number $z$,

$$z + \overline{z} = 2\Re(z).$$

So,

$$\langle u, v \rangle + \overline{\langle u, v \rangle} = 2\Re\langle u, v \rangle.$$

**Proposition 3.** *For any Hilbert space, $H$, for any $u$ and $v$ in $H$,*

$$|\langle u, v \rangle| \leq ||u||||v||.$$

**Proof:** Assume that at least one of the two is non-zero. WLOG let's assume $v \neq 0$. Let's play around with

$$||u + tv||^2 = ||u||^2 + 2t\Re\langle u, v \rangle + t^2||v||^2, \quad t \in \mathbb{R}.$$

This is a real valued function of $t$. It's a quadratic function of $t$ in fact. The derivative is

$$2t||v||^2 + 2\Re\langle u, v \rangle.$$

It's an up quadratic function, so its unique minimum is when

$$t = -\frac{\Re\langle u, v \rangle}{||v||^2}.$$

If we then check out what happens at this value of $t$,

$$||u + tv||^2 = ||u||^2 - 2\frac{\Re\langle u, v \rangle}{||v||^2}\Re\langle u, v \rangle + \Re\langle u, v \rangle^2 \frac{||v||^2}{||v||^4} = ||u||^2 - \frac{\Re\langle u, v \rangle^2}{||v||^2}.$$

We know that
$$0 \leq ||u+tv||^2$$
so we get
$$0 \leq ||u||^2 - \frac{\Re\langle u,v\rangle^2}{||v||^2} \implies 0 \leq ||u||^2||v||^2 - \Re\langle u,v\rangle^2.$$
This gives us
$$\Re\langle u,v\rangle^2 \leq ||u||^2||v||^2.$$
Well, this is annoying because of that silly $\Re$. I wonder how we could make it turn into $|\langle u,v\rangle|$? Also, we don't want to screw up the $||u||^2||v||^2$ part. Well, we know how the scalar product interacts with complex numbers, for $\lambda \in \mathbb{C}$,
$$\langle \lambda u, v\rangle = \lambda\langle u,v\rangle.$$
So, if for example
$$\langle u,v\rangle = re^{i\theta}, r = |\langle u,v\rangle| \text{ and } \theta \in \mathbb{R}.$$
We can modify u, without changing $||u||$,
$$||e^{-i\theta}u|| = ||u||.$$
Moreover
$$\langle e^{-i\theta}u, v\rangle = e^{-i\theta}\langle u,v\rangle = e^{-i\theta}re^{i\theta} = |\langle u,v\rangle|.$$
So, if we repeat everything above replacing $u$ with $e^{-i\theta}u$ we get
$$\Re\langle e^{-i\theta}u,v\rangle^2 \leq ||e^{-i\theta}u||^2||v||^2 = ||u||^2||v||^2,$$
and by the above calculation
$$\langle e^{-i\theta}u,v\rangle = |\langle u,v\rangle| \in \mathbb{R} \implies \Re\langle e^{-i\theta}u,v\rangle^2 = |\langle u,v\rangle|^2.$$
So, we have
$$|\langle u,v\rangle|^2 \leq ||u||^2||v||^2.$$
Taking the square root of both sides completes the proof of the Cauchy-Schwarz inequality.

We also have a triangle inequality.

**Proposition 4.** *For any $u$ and $v$ in a Hilbert space $H$,*
$$||u+v|| \leq ||u|| + ||v||.$$

**Proof:** We just use the previous two results:
$$||u+v||^2 = ||u||^2 + 2\Re\langle u,v\rangle + ||v||^2 \leq ||u||^2 + 2||u||||v|| + ||v||^2 = (||u|| + ||v||)^2$$
so rooting we get the triangle inequality.

**Proposition 5.** *We have the Pythagorean theorem: if $u$ and $v$ are orthogonal, then*
$$||u+v||^2 = ||u||^2 + ||v||^2.$$
*Moreover, if $\{u_n\}_{n\geq 1}$ are an ONB for the Hilbert space $H$, then for any $v \in H$,*
$$||v||^2 = \sum_{n\geq 1} |\langle v, u_n\rangle|^2.$$

**Proof:** The first statement follows from

$$||u+v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 = ||u||^2 + ||v||^2,$$

if $u$ and $v$ are orthogonal, because in that case their scalar product is zero. Moreover, for any collection of orthogonal vectors $\{u_1, \ldots, u_n\}$ we proceed by induction. Assume that

$$||u_1 + \ldots + u_{n-1}||^2 = \sum_{k=1}^{n-1} ||u_k||^2.$$

Then, if $u_n$ is orthogonal to all of $u_1, \ldots, u_{n-1}$ we also have

$$\langle u_n, u_1 + \ldots + u_{n-1}\rangle = \langle u_n, u_1\rangle + \ldots + \langle u_n, u_{n-1}\rangle = 0 + \ldots + 0.$$

Hence $u_n$ is also orthogonal to the sum,

$$\sum_{k=1}^{n-1} u_k.$$

By the Pythagorean theorem,

$$||u_n + \sum_{k=1}^{n-1} u_k||^2 = ||u_n||^2 + ||\sum_{k=1}^{n-1} u_k||^2.$$

By the induction assumption

$$= ||u_n||^2 + \sum_{k=1}^{n-1} ||u_k||^2 = \sum_{k=1}^{n} ||u_k||^2.$$

If $\{u_n\}$ are an ONB, then we can write any $v \in H$ as

$$v = \sum_{n\geq 1} c_n u_n, \quad c_n \in \mathbb{C},\ n \geq 1.$$

In other words,

$$\lim_{N\to\infty} \sum_{n=1}^{N} c_n u_n = v.$$

Hence

$$\lim_{N\to\infty} ||\sum_{n=1}^{N} c_n u_n - v|| = 0.$$

By the triangle inequality

$$||v|| = ||v - \sum_{n=1}^{N} c_n u_n + \sum_{n=1}^{N} c_n u_n|| \leq ||v - \sum_{n=1}^{N} c_n u_n|| + ||\sum_{n=1}^{N} c_n u_n||,$$

and

$$||\sum_{n=1}^{N} c_n u_n|| = ||\sum_{n=1}^{N} c_n u_n - v + v|| \leq ||\sum_{n=1}^{N} c_n u_n - v|| + ||v||.$$

Hence,

$$\left| ||v|| - ||\sum_{n=1}^{N} c_n u_n|| \right| \leq ||\sum_{n=1}^{N} c_n u_n - v||.$$

So,

$$\lim_{N\to\infty}\left| ||\sum_{n=1}^{N} c_n u_n|| - ||v|| \right| \implies \lim_{N\to\infty} ||\sum_{n=1}^{N} c_n u_n|| = ||v||,$$

and

$$\lim_{N\to\infty} ||\sum_{n=1}^{N} c_n u_n||^2 = ||v||^2.$$

By the Pythagorean theorem

$$||\sum_{n=1}^{N} c_n u_n||^2 = \sum_{n=1}^{N} ||c_n u_n||^2 = \sum_{n=1}^{N} |c_n|^2.$$

Hence, we have

$$\lim_{N\to\infty} \sum_{n=1}^{N} |c_n|^2 = \sum_{n\geq 1} |c_n|^2 = ||v||^2.$$

Finally, we note that

$$\langle v, u_m\rangle = \langle \sum_{n\geq 1} c_n u_n, u_m\rangle = \sum_{n\geq 1} c_n \langle u_n, u_m\rangle = c_m,$$

because of the assumption that $u_n$ are orthonormal.