FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2018.01.29

Forward the light brigade. As I am writing these notes, it becomes apparent that there are certain delicate issues upon which Gerald and I respectfully have differing ideas about the presentation. Hilbert spaces is one of these. I just don't like how he tries to hide the $\mathcal{L}^2$ theory under the rug. I prefer to give you the concise mathematical information leave the decision up to you: would you like to hide certain bits under the rug? Well, that's okay, but you should know for future reference where to find those bits (like this almost everywhere, sets of measure zero, Lebesgue integration, measurability stuff). It's fine to ignore it (hide it under the rug), but I think you should at least know it's under there.

Now, it turns out that when I said we *assume* the scalar product is continuous, this was not necessary. We will actually see that this is true, but we get it for free by the other parts in the definition of the norm! That's nice! This is somewhat irrelevant, because the important thing is that the scalar product is continuous. The reason *why* it's continuous is not so important. So, feel free to skip this bit if you like.

Proposition 1. *Using only the assumptions that the scalar product satisfies:*

$$\langle u, v\rangle = \overline{\langle v, u\rangle}$$

$$\langle au, v\rangle = a\langle u, v\rangle$$

$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$

$$\langle u, u\rangle \geq 0, \quad \langle u, u\rangle = 0 \iff u = 0,$$

then the scalar product is a continuous function from $H \times H \to \mathbb{C}$.

Proof: It suffices to estimate

$$|\langle u, v\rangle - \langle u', v'\rangle|.$$

I would like to somehow get

$$u - u' \text{ and } v - v'.$$

So, well, just throw them in the first and last

$$\langle u - u', v\rangle = \langle u, v\rangle - \langle u', v\rangle.$$

That shows that
$$\langle u-u', v\rangle + \langle u', v\rangle = \langle u, v\rangle.$$
So, we see that
$$\langle u, v\rangle - \langle u', v'\rangle = \langle u-u', v\rangle + \langle u', v\rangle - \langle u', v'\rangle$$
We can smash the last two terms together because $-1 \in \mathbb{R}$ so
$$-\langle u', v'\rangle = \langle u', -v'\rangle \implies \langle u', v\rangle - \langle u', v'\rangle = \langle u', v-v'\rangle.$$
Hence,
$$|\langle u, v\rangle - \langle u', v'\rangle| = |\langle u-u', v\rangle + \langle u', v-v'\rangle|.$$
By the triangle inequality
$$|\langle u-u', v\rangle + \langle u', v-v'\rangle| \leq |\langle u-u', v\rangle| + |\langle u', v-v'\rangle|.$$
By the Cauchy-Schwarz inequality
$$|\langle u-u', v\rangle| + |\langle u', v-v'\rangle| \leq ||u-u'||||v|| + ||u'||||v-v'||.$$
We therefore see that for any fixed pair $(u, v) \in H \times H$, given $\epsilon > 0$, we can define
$$\delta := \min\left\{\frac{\varepsilon}{2(||v||+1)}, \frac{\varepsilon}{2(||u||+1)}, 1\right\}.$$
Then we estimate
$$||u-u'|| < \delta \implies ||u'|| < ||u|| + \delta \leq ||u|| + 1,$$
$$||u-u'||||v|| \leq \frac{\varepsilon||v||}{2(||v||+1)} < \frac{\varepsilon}{2}.$$
and
$$||u'||||v-v'|| \leq \frac{(||u||+1)\varepsilon}{2(||u||+1)} \leq \frac{\varepsilon}{2},$$
so we obtain
$$|\langle u, v\rangle - \langle u', v'\rangle| < \varepsilon.$$

We proceed with some important theory about Hilbert spaces. This theory is going to be important because we will use Sturm-Liouville theory later to define many different (but equivalent) orthogonal bases for $\mathcal{L}^2$ on finite intervals. Why is this useful? Because we can solve PDEs in this way!!! That's the whole reason for all this theory. Here is the main idea:

(1) Start with a PDE where the x variable is in a finite (bounded) interval.
(2) Assume you can write your unknown function, u, (the unsub) as a product like $u(x,t) = X(x)T(t)$. Plug it into the PDE.
(3) Solve for X using the boundary conditions. This will probably give lots of Xs which can be indexed by $\mathbb{N}$.
(4) Each X_n has a partner T_n. Solve for these. Probably, you've got some unknown constants.

(5) Is the PDE homogeneous? If so, $X_1T_1 + X_2T_2 + \ldots$ also solves the PDE so you can smash them together into a big party series. If *not* then you may need to do something else (i.e. steady state solution). In the homogeneous case, you will then use the IC and the collection $\{X_n\}$ to find the coefficients in T_n and end up with a solution of the form

$$\sum_{n\in\mathbb{N}} X_n(x)T_n(t).$$

It's precisely in this last step where the Hilbert space theory is being used to say that you can use the X_n obtain the IC, because the Hilbert space theory tells us when certain functions are basis functions for $\mathcal{L}^2$!

So, the motivation is still to solve PDEs, even if things start to seem more abstract. Now, we've had one version of Bessel's inequality which was for the Fourier coefficients of an $\mathcal{L}^2$ function. This next version says basically the same thing, but for arbitrary Hilbert spaces, not just $\mathcal{L}^2$. Since it is super important for the proof, we include this proof here:

Proposition 2. *We have the Pythagorean theorem: if u and v are orthogonal, then*

$$||u + v||^2 = ||u||^2 + ||v||^2.$$

Moreover, if $\{u_n\}_{n\geq 1}$ are an ONB for the Hilbert space H, then for any $v \in H$,

$$||v||^2 = \sum_{n\geq 1} |\langle v, u_n\rangle|^2 .$$

Proof: The first statement follows from

$$||u + v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 = ||u||^2 + ||v||^2,$$

if u and v are orthogonal, because in that case their scalar product is zero. Moreover, for any collection of orthogonal vectors $\{u_1, \ldots, u_n\}$ we proceed by induction. Assume that

$$||u_1 + \ldots + u_{n-1}||^2 = \sum_{k=1}^{n-1} ||u_k||^2.$$

Then, if u_n is orthogonal to all of $u_1, \ldots, u_{n-1}$ we also have

$$\langle u_n, u_1 + \ldots + u_{n-1}\rangle = \langle u_n, u_1\rangle + \ldots + \langle u_n, u_{n-1}\rangle = 0 + \ldots + 0.$$

Hence u_n is also orthogonal to the sum,

$$\sum_{k=1}^{n-1} u_k.$$

By the Pythagorean theorem,

$$||u_n + \sum_{k=1}^{n-1} u_k||^2 = ||u_n||^2 + ||\sum_{k=1}^{n-1} u_k||^2.$$

By the induction assumption

$$= ||u_n||^2 + \sum_{k=1}^{n-1} ||u_k||^2 = \sum_{k=1}^{n} ||u_k||^2.$$

If $\{u_n\}$ are an ONB, then we can write any $v \in H$ as

$$v = \sum_{n \geq 1} c_n u_n, \quad c_n \in \mathbb{C},\, n \geq 1.$$

In other words,

$$\lim_{N \to \infty} \sum_{n=1}^{N} c_n u_n = v.$$

Hence

$$\lim_{N \to \infty} || \sum_{n=1}^{N} c_n u_n - v|| = 0.$$

By the triangle inequality

$$||v|| = ||v - \sum_{n=1}^{N} c_n u_n + \sum_{n=1}^{N} c_n u_n|| \leq ||v - \sum_{n=1}^{N} c_n u_n|| + ||\sum_{n=1}^{N} c_n u_n||,$$

and

$$||\sum_{n=1}^{N} c_n u_n|| = ||\sum_{n=1}^{N} c_n u_n - v + v|| \leq ||\sum_{n=1}^{N} c_n u_n - v|| + ||v||.$$

Hence,

$$\left| ||v|| - ||\sum_{n=1}^{N} c_n u_n|| \right| \leq ||\sum_{n=1}^{N} c_n u_n - v||.$$

So,

$$\lim_{N \to \infty} \left| ||\sum_{n=1}^{N} c_n u_n|| - ||v|| \right| \implies \lim_{N \to \infty} ||\sum_{n=1}^{N} c_n u_n|| = ||v||,$$

and

$$\lim_{N \to \infty} ||\sum_{n=1}^{N} c_n u_n||^2 = ||v||^2.$$

By the Pythagorean theorem

$$||\sum_{n=1}^{N} c_n u_n||^2 = \sum_{n=1}^{N} ||c_n u_n||^2 = \sum_{n=1}^{N} |c_n|^2.$$

Hence, we have

$$\lim_{N \to \infty} \sum_{n=1}^{N} |c_n|^2 = \sum_{n \geq 1} |c_n|^2 = ||v||^2.$$

Finally, we note that

$$\langle v, u_m \rangle = \langle \sum_{n \geq 1} c_n u_n, u_m \rangle = \sum_{n \geq 1} c_n \langle u_n, u_m \rangle = c_m,$$

because of the assumption that u_n are orthonormal.

Theorem 3 (Bessel's Inequality for general Hilbert spaces)**.** *Let $\{\phi_n\}_{n \in \mathbb{N}}$ be an orthonormal set in a Hilbert space H. Then if $f \in H$,*

$$\sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle| \leq ||f||^2.$$

Proof: By the Pythagorean theorem, for each $N \in \mathbb{N}$,

$$\| \sum_{n=1}^{N} \hat{f}_n \phi_n \|^2 = \sum_{n=1}^{N} |\hat{f}_n|^2.$$

Above, we have used the convenient notation

$$\hat{f}_n = \langle f, \phi_n \rangle.$$

So, we compute that the square of the distance between f and its partial Fourier series

$$0 \leq \|f - \sum_{n=1}^{N} \hat{f}_n \phi_n\|^2 = \|f\|^2 - 2\Re \langle f, \sum_{1}^{N} \hat{f}_n \phi_n \rangle + \| \sum_{1}^{N} \hat{f}_n \phi_n \|^2.$$

Let's look at the middle bit:

$$\langle f, \sum_{1}^{N} \hat{f}_n \phi_n \rangle = \sum_{1}^{N} \overline{\hat{f}_n} \langle f, \phi_n \rangle = \sum_{1}^{N} \overline{\hat{f}_n} \hat{f}_n = \sum_{1}^{n} |\hat{f}_n|^2.$$

Hence,

$$0 \leq ||f||^2 - 2 \sum_{1}^{N} |\hat{f}_n|^2 + \sum_{n=1}^{N} |\hat{f}_n|^2 = ||f||^2 - \sum_{1}^{N} |\hat{f}_n|^2$$

so re-arranging

$$\sum_{1}^{N} |\hat{f}_n|^2 \leq ||f||^2.$$

Letting $N \to \infty$ completes the proof.

Now, it turns out that the version of Bessel's inequality for the Fourier coefficients will actually be an equality, because $\{e^{inx}\}_{n \in \mathbb{Z}}$ is a basis for $\mathcal{L}^2$ on $[-\pi, \pi]$. In general, Bessel's inequality on a Hilbert space becomes an equality if and only if the orthonormal set $\{\phi_n\}$ is a basis.

1.1. **Proof of the 3 equivalent conditions to be an ONB in a Hilbert space.**

This seems to be a fun one for some reason. It is rather nicely straightforward. Perhaps what makes it so nice is the pleasant setting of a Hilbert space, or translated directly from German, a Hilbert room. Hilbert rooms are cozy.

Theorem 4. *Let $\{\phi_n\}_{n \in \mathbb{N}}$ be orthonormal in a Hilbert space, H. TFAE (the following are equivalent):*

$$(1) \quad f \in H \ \textit{och} \ \langle f, \phi_n \rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 .$$

Proof: We shall proceed in order prove (1) $\implies$ (2), then (2) $\implies$ (3), and finally (3) $\implies$ (1). Just stay calm and carry on. So we begin by assuming (1) holds, and then we shall show that (2) must hold as well. First, we note that by Bessel's inequality, the series

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \le ||f||^2 < \infty.$$

Hence, if we know anything about convergent series, then we sure better know that the tail of the series tends to zero. The tail of the series is

$$\sum_{n\ge N} |\langle f, \phi_n\rangle|^2 \to 0 \text{ as } N \to \infty.$$

Now, let us define some elements in our Hilbert space, which we shall show comprise a Cauchy sequence. Let

$$g_N := \sum_{n=1}^{N} \langle f, \phi_n\rangle \phi_n.$$

For $M \ge N$, we have, using the Pythagorean Theorem and the orthonormality of the $\{\phi_n\}$,

$$||g_M - g_N||^2 = || \sum_{n=N+1}^{M} \langle f, \phi_n\rangle\phi_n||^2 = \sum_{n=N+1}^{M} |\langle f, \phi_n\rangle|^2 \le \sum_{n=N+1}^{\infty} |\langle f, \phi_n\rangle|^2 \to 0 \text{ as } N \to \infty.$$

Hence, by definition of Cauchy sequence (which one really should know at this point!), $\{g_N\}_{N\ge 1}$ is a Cauchy sequence in our Hilbert space. By definition of Hilbert space, every Hilbert space is complete. Thus every Cauchy sequence converges to a unique limit. Let us now call the limit of our Cauchy sequence, which is by definition,

$$\lim_{N\to\infty} g_N = \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, \phi_n\rangle\phi_n = \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle\phi_n = g.$$

We will now show that $f - g$ satisfies

$$\langle f - g, \phi_n\rangle = 0 \forall n \in \mathbb{N}.$$

Then, because we are assuming (1) holds, this implies that $f - g = 0$, ergo $f = g$. So, we compute this inner product,

$$\langle f - g, \phi_n\rangle = \langle f, \phi_n\rangle - \langle g, \phi_n\rangle.$$

We insert the definition of g as the series,

$$\langle g, \phi_n\rangle = \langle \sum_{m\ge 1} \langle f, \phi_m\rangle\phi_m, \phi_n\rangle = \sum_{m\ge 1} \langle f, \phi_m\rangle\langle\phi_m, \phi_n\rangle = \langle f, \phi_n\rangle.$$

Above, we have used in the second equality the linearity of the inner product and the continuity of the inner product. In the third equality, we have used that $\langle\phi_m, \phi_n\rangle$ is 0 if $m \ne n$, and is 1 if $m = n$. Hence, only the term with $m = n$ survives in the sum. Thus,

$$\langle f - g, \phi_n\rangle = \langle f, \phi_n\rangle - \langle g, \phi_n\rangle = \langle f, \phi_n\rangle - \langle f, \phi_n\rangle = 0, \quad \forall n \in \mathbb{N}.$$

By (1), this shows that $f - g = 0 \implies f = g$.

Next, we shall assume that (2) holds, and we shall use this to demonstrate (3). Well, note that

$$f = \lim_{N\to\infty} g_N \implies ||f-g_N||^2 \to 0, \quad \text{as } N \to \infty.$$

Then, by the triangle inequality,

$$||f||^2 = ||f-g_N+g_N||^2 \le ||f-g_N||^2+||g_N||^2 = ||f-g_N||^2+\sum_{n=1}^{N} |\langle f,\phi_n\rangle|^2 \le |f-g_N||^2+\sum_{n\in\mathbb{N}} |\langle f,\phi_n\rangle|^2.$$

On the other hand, by Bessel's Inequality,

$$\sum_{n\in\mathbb{N}} |\langle f,\phi_n\rangle|^2 \le ||f||^2.$$

So, we have a little sandwich, en smörgås, if you will, with $||f||^2$ right in the middle of our sandwich,

$$\sum_{n\in\mathbb{N}} |\langle f,\phi_n\rangle|^2 \le ||f||^2 \le ||f-g_N||^2 + \sum_{n\in\mathbb{N}} |\langle f,\phi_n\rangle|^2.$$

Letting $N\to\infty$ on the right side, the term $||f-g_N||\to 0$, and so we indeed have

$$\sum_{n\in\mathbb{N}} |\langle f,\phi_n\rangle|^2 \le ||f||^2 \le \sum_{n\in\mathbb{N}} |\langle f,\phi_n\rangle|^2.$$

This of course means that all three terms are equal, because the terms all the way on the left and right side are the same!

Finally, we assume (3) holds and use it to show that (1) must also hold. This is pleasantly straightforward. We assume that for some f in our Hilbert space, $\langle f,\phi_n\rangle = 0$ for all n. Using (3), we compute

$$||f||^2 = \sum_{n\in\mathbb{N}} |\langle f,\phi_n\rangle|^2 = \sum_{n\in N} 0 = 0.$$

The only element in a Hilbert space with norm equal to zero is the 0 element. Thus $f=0$.