FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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The Hilbert space theory is useful for solving PDEs. We will see this connection through Sturm-Liouville problems. We've just got a little bit of theory to complete before we get to the SLPs.

1.1. The Best Approximation Theorem.

This is another fun and cozy Hilbert room theory item.

Theorem 1. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space, H. If $f \in H$, then*

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

and equality holds $\iff$ $c_n = \langle f, \phi_n\rangle$ is true $\forall n \in \mathbb{N}$.

Proof: We make a few definitions: let

$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$

and

$$\varphi := \sum c_n \phi_n.$$

Then we compute

$$||f-\varphi||^2 = ||f-g+g-\varphi||^2 = ||f-g||^2 + ||g-\varphi||^2 + 2\Re\langle f-g, g-\varphi\rangle.$$

I claim that

$$\langle f-g, g-\varphi\rangle = 0.$$

Just write it out (stay calm and carry on):

$$\langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$

$$= \sum \overline{\widehat{f_n}}\langle f, \phi_n\rangle - \sum \overline{c_n}\langle f, \phi_n\rangle - \sum \widehat{f_n}\langle \phi_n, \sum \widehat{f_m}\phi_m\rangle + \sum \widehat{f_n}\langle \phi_n, \sum c_m\phi_m\rangle$$

$$= \sum |\widehat{f_n}|^2 - \sum \overline{c_n}\widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n}\overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have

$$||f-\varphi||^2 = ||f-g||^2 + ||g-\varphi||^2 \geq ||f-g||^2,$$

with equality iff

$$||g - \varphi||^2 = 0.$$

Let us now write out what this norm is, using the definitions of g and φ. By their definitions,

$$g - \varphi = \sum (\widehat{f_n} - c_n)\phi_n.$$

By the Pythagorean theorem, due to the fact that the ϕ_n are an orthonormal set, and hence multiplying them by the scalars, $\widehat{f_n} - c_n$, they remain orthogonal, we have

$$||g - \varphi||^2 = \sum \left|\widehat{f_n} - c_n\right|^2.$$

This is a sum of non-negative terms. Hence, the sum is only zero if all of the terms in the sum are zero. The terms in the sum are all zero iff

$$\left|\widehat{f_n} - c_n\right| = 0 \forall n \iff c_n = \widehat{f_n} \forall n \in \mathbb{N}.$$

1.2. **Spectral Theorem Motivation.** Basically, a linear (partial or ordinary) differential operator with constant coefficients will act on a certain Hilbert space. For example, the operator

$$\Delta = -\partial_x^2$$

acts on the Hilbert space H^2. Don't worry about what it is precisely, because what's important is just that it's a Hilbert space. This operator takes elements of the Hilbert space H^2[1] and sends them to the Hilbert space $\mathcal{L}^2$. It is a linear operator because

$$\partial_x^2(f(x) + g(x)) = f''(x) + g''(x) = \partial_x^2(f(x)) + \partial_x^2(g(x)).$$

So if we think of the functions as vectors, then Δ is like a linear map that takes in vectors and spits out vectors. Just like linear maps on finite dimensional vector spaces, which can be represented by a matrix, a linear operator on a Hilbert space can be represented by a matrix. If it is a sufficiently "nice" operator, then there will exist an orthonormal basis of eigenfunctions with corresponding eigenvalues. Here it is useful to recall

Theorem 2 (Spectral Theorem for $\mathbb{C}^n$)**.** *Assume that A is a Hermitian matrix. Then there exists an orthonormal basis of $\mathbb{C}^n$ which consists of eigenvectors of A. Moreover, each of the eigenvalues is real.*

Proof: Remember what Hermitian means. It means that for any $u, v \in \mathbb{C}^n$, we have

$$\langle Au, v\rangle = \langle u, Av\rangle.$$

By the Fundamental Theorem of Algebra, the characteristic polynomial

$$p(x) := \det(A - xI)$$

factors over $\mathbb{C}$. The roots of p are $\{\lambda_k\}_{k=1}^n$. These are by definition the eigenvalues of A. First, we consider in case A has zero as an eigenvalue. If this is the case, then we define

$$K_0 := \mathrm{Ker}(A) = \{u \in \mathbb{C}^n : Au = 0\}.$$

[1]The Hilbert space, H^2 sits inside $\mathcal{L}^2$.

We note that all nonzero $u \in K_0$ are eigenvectors of A for the eigenvalue 0. Since K_0 is a k-dimensional subspace of $\mathbb{C}^n$, it has an ONB $\{v_1, \ldots, v_k\}$. If $k = n$, we are done. So, assume that $k < n$. Then we consider

$$K_0^\perp = \{u \in \mathbb{C}^n : \langle u, v \rangle = 0 \forall v \in K_0\}.$$

Any $u \in \mathbb{C}^n$ can the be written as

$$u = u_0 + u_0^\perp, \quad u_0 \in K_0 \text{ and } u_0^\perp \in K_0^\perp.$$

Since A has eigenvalues $\{\lambda_j\}_{j=1}^n$, and 0 appears with multiplicity k, $\lambda_{k+1} \neq 0$. It has some non-zero eigenvector. Let's call it u. Then we compute

$$\langle Au, v \rangle = \lambda_{k+1} \langle u, v \rangle = \langle u, Av \rangle = 0 \forall v \in K_0.$$

Hence, we see that

$$u \in K_0^\perp.$$

Since it is an eigenvector it is not zero, so we define

$$v_{k+1} := \frac{u}{||u||}.$$

Next we define K_1 to be the span of the vectors $\{v_1, \ldots, v_{k+1}\}$. We look at A restricted to $K_1^\perp$. We note that A maps K_1 to itself because if

$$v = \sum_1^{k+1} c_j v_j \implies Av = \sum_1^{k+1} c_j A v_j = \sum_1^{k+1} c_j \lambda_j v_j \in K_1.$$

Similarly, if $w \in K_1^\perp$,

$$\langle Aw, v \rangle = \langle w, Av \rangle = 0 \forall v \in K_1.$$

So, A maps $K_1^\perp$ into itself. Since the kernel of A is in K_1, A is a surjective and injective map from $K_1^\perp$ into itself. So, there is an eigenvalue λ_{k+2}, for A as a linear map from $K_1^\perp$ to itself. It has an eigenvector, which we may assume has unit length, contained in $K_1^\perp$. Call it v_{k+2}. Continue inductively until we reach in this way $\{v_1, \ldots, v_n\}$ to span $\mathbb{C}^n$.

Let us do an example. On $[-\pi, \pi]$, the functions which satisfy

$$\Delta f = \lambda f, \quad f(-\pi) = f(\pi)$$

are

$$f(x) = f_n(x) = e^{inx}.$$

The corresponding

$$\lambda_n = n^2.$$

So, the eigenvalues of Δ with this particular boundary condition are n^2, and the corresponding eigenfunctions are $e^{\pm inx}$. We have proven that these are orthogonal. We can make them orthonormal by dividing by the norms,

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}.$$

We note that for all f and g in $\mathcal{L}^2$ which satisfy $f(-\pi) = f(\pi)$, $g(-\pi) = g(\pi)$ and which are also (at least weakly) twice differentiable, we would also get $f'(-\pi) = f'(\pi)$ and similarly for g, so that

$$\langle \Delta f, g\rangle = \int_{-\pi}^{\pi} -f''(x)\overline{g(x)}dx = -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx$$
$$= -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + f(x)\overline{g'(x)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx.$$

Due to the boundary conditions, all that survives is

$$-\int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx = \langle f, \Delta g\rangle.$$

So we see that

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle.$$

This is just like the spectral theorem for Hermitian matrices! There is a similar spectral theorem here, a "grown-up linear algebra" theorem, called The Spectral Theorem. This grown-up version of the spectral theorem says that, like a Hermitian matrix, the operator Δ also has an $\mathcal{L}^2$ orthonormal basis of eigenfunctions. Hence, by this theorem, we know that the orthonormal *set,*

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}},$$

(which à priori could be missing stuff) is in fact not missing anything, spans all of $\mathcal{L}^2$, and is an ONB. If you're interested in this topic, you can try to convince me to give a PhD/Master's course on it. With sufficiently many interested students, I may be convinced.

1.3. **Regular SLPs.** Let L be a linear, second order ordinary differential operator. So, we can write

$$L(f) = r(x)f''(x) + q(x)f'(x) + p(x)f(x).$$

Above, r, q, and p are specified REAL VALUED functions. As a simple example, take $r(x) = -1$, and $q(x) = p(x) = 0$. Then we have

$$L(f) = \Delta f = -f''(x).$$

We are working with functions defined on an interval $[a, b]$ which is a *finite* interval. So, the Hilbert space in which everything is happening is $\mathcal{L}^2$ on that interval. Like with matrices, we can think about the *adjoint* of the operator L. The adjoint by definition satisfies

$$\langle Lf, g\rangle = \langle f, L * g\rangle,$$

where we are using L^* to denote the adjoint operator. Whatever it is. On the left side, we know what everything is, so we write it out by definition of the scalar product

$$\langle Lf, g\rangle = \int_a^b L(f)\overline{g(x)}dx = \int_a^b (r(x)f''(x) + q(x)f'(x) + p(x)f(x))\,\overline{g(x)}dx.$$

Integrating by parts, we get

$$= (r\bar{g})f'\big|_a^b - \int_a^b (r\bar{g})'f' + (qg)f\big|_a^b - \int_a^b (q\bar{g})'f + \int_a^b pf\bar{g}$$

$$= (r\bar{g})f' + (q\bar{g})f\big|_a^b - \int_a^b [(r\bar{g})'f' + (q\bar{g})'f - pf\bar{g}].$$

We integrate by parts once more on the $(r\bar{g})'f'$ term to get

$$= (r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)\big|_a^b + \int_a^b (r\bar{g})''f - (q\bar{g})'f + fp\bar{g}.$$

So, if the boundary conditions are chosen to make the stuff evaluated from a to b (these are called the boundary terms in integration by parts) vanish, then we could define

$$L^*g = (rg)'' - (qg)' + pg,$$

since then

$$\langle Lf, g\rangle = \int_a^b (r\bar{g})''f - (q\bar{g})'f + fp\bar{g} = \langle f, L^*g\rangle.$$

Here we use that r, q and p are real valued functions, so $\bar{r} = r$, $\bar{q} = q$, and $\bar{p} = p$. For the spectral theorem to work, we will want to have

$$L = L^*.$$

When this holds, we say that L *is formally self-adjoint.* So, we need

$$Lf = L^*f \iff rf'' + qf' + pf = (rf)'' - (qf)' + pf.$$

We write the things out:

$$rf''+qf'+pf = (rf'+r'f)'-qf'-q'f+pf \iff rf''+qf' = rf''+2r'f'+r''f-qf'-q'f$$

$$\iff qf' = 2r'f' + r''f - qf' - q'f \iff (2q - 2r')f' + (q' - r'')f = 0.$$

To ensure this holds for all f, we set the coefficient functions equal to zero:

$$2q - 2r' = 0 \implies q = r', \quad q' = r''.$$

Well, that just means that $q = r'$. So, we need L to be of the form

$$Lf = rf'' + r'f' + pf = (rf')' + pf.$$

We then note that the boundary conditions we will want shall make this:

$$(r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)\big|_a^b = (r\bar{g})f' - (r\bar{g})'f + (r'\bar{g})f\big|_a^b = 0,$$

$$\iff r\bar{g}f' - r'\bar{g}f - r\bar{g}'f + r'\bar{g}f\big|_a^b = 0 \iff r\bar{g}f' - r\bar{g}'f\big|_a^b = 0$$

$$\iff r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

Hence, it is enough to have

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) - (\bar{g}(a)f'(a) - \bar{g}'(a)f(a)) = 0 \iff$$

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) = \bar{g}(a)f'(a) - \bar{g}'(a)f(a).$$

This is how we get to the definition of a regular SLP on an interval $[a, b]$. It is specified by

(1) a formally self-adjoint operator

$$L(f) = (rf')' + pf,$$

where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:
$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f(b) = 0, \quad i = 1, 2.$$
The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions
$$r(\bar{g}f' - \bar{g}'f)|_a^b = 0.$$
(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP
$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$
The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition.

We then have a miraculous fact.

Theorem 3 (Adult Spectral Theorem)**.** *For every regular Sturm-Liouville problem as above, there is an orthonormal basis of L^2_w consisting of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. We have*
$$\lim_{n\to\infty} \lambda_n = \infty.$$
Here, L^2_w is the weighted Hilbert space consisting of (the almost everywhere-equivalence classes of measurable) functions on the interval $[a, b]$ which satisfy
$$\int_a^b |f(x)|^2 w(x) dx < \infty,$$
and the scalar product is
$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)} w(x) dx.$$

We are not equipped to prove this fact. You can rest assured however that it is done through the techniques of functional analysis and bears similarity to the proof of the spectral theorem for finite dimensional vector spaces. I may be convinced to do a course which proves this fact and which also proves *how* the eigenvalues λ_n tend to ∞ as $n \to \infty$. I mean, they could go like $\lambda_n \sim n$, or like $\lambda_n \sim \sqrt{n}$ or like $\lambda_n \sim e^n$, which is it? The answer is a form of Weyl's law...