FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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We have another lovely fact which we shall prove. After that we're going to focus hard on examples of solving PDEs using the collection of techniques and theory we have amassed thus far, and we will also add a few more tricks to our bag. Recall the definition of a regular SLP:

(1) a formally self-adjoint operator

$$L(f) = (rf')' + pf,$$

where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f(b) = 0, \quad i = 1, 2.$$

The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions

$$r(\bar{g}f' - \bar{g}'f)\Big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP

$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$

The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition.

Theorem 1 (Cute facts about SLPs)**.** *Let f and g be eigenfunctions for a regular SLP in an interval $[a, b]$ with $w \equiv 1$. Let λ be the eigenvalue for f and μ the eigenvalue for g. Then:*

(1) $\lambda \in \mathbb{R}$ och $\mu \in \mathbb{R}$;

(2) If $\lambda \neq \mu$, then:

$$\int_a^b f(x)\overline{g(x)}dx = 0.$$

Proof: By definition we have $Lf + \lambda f = 0$. Moreover, L is self-adjoint, so we have

$$\langle Lf, f \rangle = \langle f, Lf \rangle.$$

By definition,

$$\langle Lf, f \rangle = \int_a^b L(f)(x)\overline{f(x)}dx.$$

Thus, we have

$$-\lambda \int_a^b |f(x)|^2 dx = -\overline{\lambda} \int_a^b |f(x)|^2 dx \iff \lambda = \overline{\lambda}.$$

The last statement holds because

$$\int_a^b |f(x)|^2 dx = ||f||_{L^2} = 0 \iff f \equiv 0,$$

and by definition of f as an eigenfunction, $f \not\equiv 0$. Same proof holds for μ.

For the second part, we use basically the same argument based on self-adjointness:

$$\langle Lf, g \rangle = \langle f, Lg \rangle.$$

By assumption

$$\langle Lf, g \rangle = -\lambda \langle f, g \rangle = \langle f, Lg \rangle = \langle f, -\mu g \rangle = -\overline{\mu} \langle f, g \rangle = -\mu \langle f, g \rangle.$$

Thus, if $\langle f, g \rangle \neq 0$, this forces $\lambda = \mu$, which is false. Hence, the only viable option is that $\langle f, g \rangle = 0$. By definition,

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

1.0.1. *SLP example.* Here is how the SLP theory can be useful in practice. We are given the problem

$$u_t - u_{xx} = 0, \quad u_x(0,t) = \alpha u(0,t), \quad u_x(l,t) = -\alpha u(l,t), \quad u(x,0) = f(x).$$

Above, we assume that

$$\alpha > 0, \quad f \in \mathcal{L}^2.$$

These BCs come from Newton's law of cooling: the temperature gradient across the ends is proportional to the temperature difference between the ends and the surrounding medium. It is a homogeneous PDE, so we have good chances of being able to solve it using separation of variables. Thus, we write

$$u(x,t) = X(x)T(t) \implies T'(t)X(x) - X''(x)T(t) = 0 \implies \frac{T'}{T} = \frac{X''}{X}.$$

This means both sides are equal to a constant. Call it λ. We start with the x side, because we have more information about that due to the BCs. Are they self-adjoint BCs? Let's check! In the definition of SLP, we are looking for X to satisfy

$$\frac{X''}{X} = \lambda \iff X'' = \lambda X \iff X'' - \lambda X = 0.$$

Thus we see that the r and w are both 1 in the definition of SLP, and the p is 0. The $a = 0$ and $b = l$. So, we need to check that if f and g satisfy

$$f'(0) = \alpha f(0), \quad g'(l) = -\alpha g(l)$$

then
$$(\bar{g}f' - \bar{g}'f)\big|_0^l = 0.$$
We plug it in
$$\bar{g}(l)f'(l) - \bar{g}'(l)f(l) - \bar{g}(0)f'(0) + \bar{g}'(0)f(0)$$
$$= -\bar{g}(l)\alpha f(l) + \alpha \bar{g}(l)f(l) - \bar{g}(0)\alpha f(0) + \alpha \bar{g}(0)f(0) = 0.$$
Yes, the BC is a self-adjoint BC. So, the SLP theorem says there exists an $\mathcal{L}^2$ ONB of eigenfunctions. What are they? We check the cases.
$$X'' = \lambda X.$$
What if $\lambda = 0$? Then
$$X(x) = ax + b.$$
To get
$$X'(0) = \alpha X(0) \implies a = \alpha b \implies b = \frac{a}{\alpha}.$$
Next,
$$X'(l) = -\alpha X(l) \implies a = -\alpha\left(al + \frac{a}{\alpha}\right) = -a(\alpha l + 1).$$
Presumably $a \neq 0$ because if $a = 0$ the whole solution is just 0. So, we can divide by it and we get
$$\implies 1 = -(\alpha l + 1) \implies \alpha l = -2.$$
Since $l > 0$ and $\alpha > 0$, this is impossible. So, no solutions for $\lambda = 0$.

Next we try $\lambda > 0$. Then the solution looks like
$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$$
or equivalently, we can use sinh and cosh, to write
$$X(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x).$$
We try out the BCs. They require
$$X'(0) = \alpha X(0) \iff a\sqrt{\lambda}\sinh(0) + b\sqrt{\lambda}\cosh(0) = \alpha\left(a\cosh(0) + b\sinh(0)\right)$$
$$\iff b\sqrt{\lambda} = \alpha a \implies b = \frac{\alpha a}{\sqrt{\lambda}}.$$
We check out the other BC:
$$X'(l) = -\alpha X(l) \iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l) + \alpha a\cosh(\sqrt{\lambda}l) = -\alpha\left(a\cosh(\sqrt{\lambda}l) + \frac{\alpha a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l)\right).$$
$$\iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l) + \frac{\alpha^2 a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l) = -2\alpha a\cosh(\sqrt{\lambda}l)$$
If $a = 0$ the whole solution is zero, so we presume that is not the case and divide by a. Then this requires
$$\frac{\sinh(\sqrt{\lambda}l)}{\cosh(\sqrt{\lambda}l)} = \frac{-2\alpha}{\sqrt{\lambda} + \alpha^2/\sqrt{\lambda}}.$$
Equivalently
$$\tanh(\sqrt{\lambda}l) = \frac{-2\alpha\sqrt{\lambda}}{\lambda + \alpha^2}.$$
Are there solutions to this equation? Well, $\sqrt{\lambda}l > 0$. So the left side is positive, but the right side is negative. So, this equation has no solutions.

Thus, we finally try $\lambda < 0$. Then the solution looks like

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To get

$$X'(0) = \alpha X(0) \implies b\sqrt{|\lambda|} = \alpha a \implies b = \frac{\alpha a}{\sqrt{|\lambda|}}.$$

Next we need

$$X'(l) = -\alpha X(l)$$

$$\implies -a\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}l) + \frac{\alpha a}{\sqrt{|\lambda|}}\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}l) = -\alpha\left(a\cos(\sqrt{|\lambda|}l) + \frac{\alpha a}{\sqrt{|\lambda|}}\sin(\sqrt{|\lambda|}l)\right).$$

Presumably $a \neq 0$ because if that is the case then the whole solution is 0. So, we may divide by a, and we need

$$2\alpha\cos\sqrt{|\lambda|} = \sin(\sqrt{|\lambda|}l)\left(\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}\right).$$

This is equivalent to

$$\frac{2\alpha}{\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}} = \tan(\sqrt{|\lambda|}l)$$

$$\iff \frac{2\alpha\sqrt{|\lambda|}}{|\lambda| - \alpha^2} = \tan(\sqrt{|\lambda|}l).$$

Well, that's pretty weird, but according to the SLP theory, there exists a sequence

$$\{\lambda_n\}_{n\geq 1} \text{ and } \{X_n(x)\}_{n\geq 1}, \quad X_n(x) = a_n\left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}x)\right)$$

of eigenvalues and a corresponding $\mathcal{L}^2$ ONB of eigenfunctions. The coefficients a_n are defined to make

$$||X_n|| = \int_0^l |X_n(x)|^2 dx = 1.$$

To precisely specify these, it is enough to define

$$a_n := \left(\int_0^l \left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}x)\right)^2 dx\right)^{-\frac{1}{2}}.$$

That's right, I don't require you to actually do this computation in practice. It suffices to know what the coefficient should be in order to make the eigenfunctions have unit $\mathcal{L}^2$ norm. The partner functions

$$T_n(t) \text{ satisfy } T_n'(t) = \lambda_n T_n(t) \implies T_n(t) = e^{\lambda_n t}.$$

Here it is good to note that the $\lambda_n < 0$ and the tend to $-\infty$ as $n \to \infty$ which follows from the big magical theorem on SLPs. So, for heat, that is realistic. We build the solution by defining

$$u(x,t) = \sum_{n\geq 1} \hat{f}_n T_n(t) X_n(x).$$

The coefficients are just the Fourier coefficients of f with respect to the $\mathcal{L}^2$ ONB $\{X_n\}$ so

$$\hat{f}_n = \langle f, X_n \rangle = \int_0^l f(x)\overline{X_n(x)}dx.$$

1.1. **PDEs.** The screwy problem on the exam actually is useful for introducing a new tool for solving PDEs. We begin by solving

1.1.1. *IVP for homogeneous heat equation on an interval.* We take for example the problem:

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

$$u(-\pi) = u(\pi) = 0$$

$$u_t(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0.$$

We examine its characteristics.

(1) Homogeneous PDE
(2) Dirichlet BC which is a self-adjoint BC for the u_{xx} part of the equation (SLP style)
(3) Initial data which is $\mathcal{L}^2$ (continuous in fact).

Since we see these characteristics, we proceed to solve using

- Separation of variables
- SLP to get $\mathcal{L}^2$ ONB of eigenfunctions to $X_n'' = \lambda_n X_n$. Theory says the X_n are an ONB. The X_n also tell us the partner function T_n.
- IC to get the coefficients for the $u_n = X_n T_n$.
- Solution is then
$$u(x,t) = \sum u_n(x,t).$$

So - LET'S DO IT!

We begin by doing separation of variables. Write $u(x,t) = X(x)T(t)$. We get the equation

$$T'(t)X(x) - X''(x)T(t) = 0 \iff \frac{T'}{T} = \frac{X''}{X} = \lambda.$$

Since we have super nice BCs for X, we start with the X. We want to solve

$$X''(x) = \lambda X(x), \quad X(-\pi) = X(\pi) = 0.$$

First case: $\lambda = 0$. Then

$$X(x) = ax + b.$$

The BCs say

$$X(-\pi) = -a\pi + b = 0 \implies a\pi = b.$$

Next we need

$$X(\pi) = a\pi + b = 0 \implies b = -a\pi.$$

Combining these,

$$a\pi = -a\pi \implies a = 0 \implies b = 0.$$

So, no solution here because the zero solution doesn't count! Moving right along, let us try

$$\lambda > 0.$$

Then, our solution looks like real exponentials or equivalently sinh and cosh. **HINT:** If you interval looks like $[0,l]$, it's probably easiest to work with sinh and cosh because $\sinh(0) = 0$ and $\cosh' = \sinh$. So this will often make things simpler. On the other hand, if you have an interval like $[a,b]$ with a and be not zero, it may be easier to work with the exponentials. So, that's why I'm choosing to do that here. Hence

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

The BCs require

$$X(-\pi) = ae^{-\sqrt{\lambda}\pi} + be^{\sqrt{\lambda}\pi} = 0.$$

Let's multiply by $e^{\sqrt{\lambda}\pi}$, to get

$$a + be^{2\sqrt{\lambda}\pi} = 0 \implies a = -be^{2\sqrt{\lambda}\pi}.$$

We check the other BCs

$$X(\pi) = ae^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0$$

substituting the value of a,

$$-be^{2\sqrt{\lambda}\pi}e^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0.$$

If $b = 0$ the whole solution is 0, so we assume this is not the case and divide by b. Multiplying by $e^{\sqrt{\lambda}\pi}$ we get

$$-e^{4\sqrt{\lambda}\pi} + 1 = 0 \iff e^{4\sqrt{\lambda}\pi} = 1 \iff 4\sqrt{\lambda}\pi = 0 \iff \lambda = 0,$$

which is a contradiction. So, no solutions lurking over here.

Thus, we consider $\lambda < 0$. Then our solution looks like

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

We need

$$X(-\pi) = a\cos(-\sqrt{|\lambda|}\pi) + b\sin(-\pi\sqrt{|\lambda|}) = 0 = a\cos(\sqrt{|\lambda|}\pi) - b\sin(\sqrt{|\lambda|}\pi),$$

where we use the evenness of cosine and oddness of sine. We also need

$$X(\pi) = a\cos(\sqrt{|\lambda|}\pi) + b\sin(\sqrt{|\lambda|}\pi) = 0.$$

Adding these equations we see that we need

$$a\cos(\sqrt{|\lambda|}\pi) = 0 \implies a = 0 \text{ or } \sqrt{|\lambda|} = \frac{(2k+1)}{2}, \quad k \in \mathbb{Z}.$$

Subtracting these equations we see that we need

$$b\cos(\sqrt{|\lambda|}\pi) = 0 \implies b = 0 \text{ or } \sqrt{|\lambda|} = \frac{2k}{2}, \quad k \in \mathbb{Z}.$$

I know it looks weird but I wrote it this way to make it looks similar to the one with the cosine. Now, the number $\sqrt{|\lambda|}$ can only have one value. It cannot be two different things at the same time. So, we have two types of solutions

$$X_n(x) = a_n\cos(\sqrt{|\lambda_n|}x) + b_n\sin(\sqrt{|\lambda_n|}x),$$

$$a_n = \begin{cases} 0 & n = \text{ even} \\ \left(\int_{-\pi}^{\pi}\cos(\sqrt{|\lambda_n|}x)^2dx\right)^{-\frac{1}{2}} & n = \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & n = \text{ odd} \\ \left(\int_{-\pi}^{\pi} \sin(\sqrt{|\lambda_n|}x)^2 dx\right)^{-\frac{1}{2}} & n = \text{ even} \end{cases}$$

with

$$\sqrt{|\lambda_n|} = \frac{n}{2}, \quad \lambda_n = -\frac{n^2}{4}.$$

I leave it as an **exercise** to compute that

$$\int_{-\pi}^{\pi} \cos(\sqrt{|\lambda_n|}x)^2 dx = \pi = \int_{-\pi}^{\pi} \sin(\sqrt{|\lambda_n|}x)^2 dx.$$

This is really just for fun. As before, it suffices to define the coefficients as given above though.

Now we solve for the partner functions,

$$\frac{T_n'}{T_n} = \lambda_n \implies T_n(t) = e^{\lambda_n t}.$$

We ignore the constant factors because they come in at the end. Normalizing the eigenfunctions X_n of the SLP, we define

$$\phi_n(x) := \frac{X_n(x)}{\sqrt{\pi}}.$$

Then, we write

$$u(x,t) = \sum_{n \in \mathbb{N}} T_n(t)\phi_n(x)\hat{v}_n,$$

where

$$v(x) := u(x,0)$$

and

$$\hat{v}_n = \langle v, \phi_n \rangle = \int_{-\pi}^{\pi} v(x)\phi_n(x)dx.$$

You can compute these Fourier coefficients if you want to do it, but it's not actually necessary to do it on the exam. Just a friendly little tip for saving time.

1.1.2. *IVP for homogeneous wave equation on an interval.* This works in a very similar way with basically the same steps as above. Please see the example from the lecture notes from Day 4 where such an example is done.