Fourieranalys MVE030 och Fourier Metoder MVE290 28.augusti.2018
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara en ortonormal mängd i ett Hilbert-rum, H. Om $f \in H$, bevisa att gäller:

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

och $=$ gäller $\iff$ $c_n = \langle f, \phi_n\rangle$ gäller $\forall n \in \mathbb{N}$. (10 p)

The solution is in the proofs of the theory items!

2. Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum, H. Bevisa att följande tre är ekvivalenta:

$$(1) \quad f \in H \text{ och } \langle f, \phi_n\rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

(10 p)

The solution is in the proofs of the theory items!

3. Beräkna den komplexa Fourierserien till den 2π-periodiska funktion $f(x)$ som är lika med x^3 i $(-\pi, \pi)$. Vad är seriens summa i punkten 8π? (10 p)

This is a two-parter. First part (5 points) is to compute the Fourier series. So, we need to compute the integrals:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 e^{-inx} dx.$$

Perhaps beta can help with this calculation? Or we just fall in love with integration by parts and keep on doing it... The idea is that each time the e^{-inx} part doesn't get any worse, but the x^3 loses a power by taking the derivative. The function whose derivative is e^{-inx} is

$$\frac{e^{-inx}}{-in}.$$

So one time IP gives:

$$\frac{1}{2\pi}\left[\left(x^3\frac{e^{-inx}}{-in}\right)_{x=-\pi}^{\pi} - \int_{-\pi}^{\pi} 3x^2\frac{e^{-inx}}{-in}dx\right].$$

Next we use the same idea to compute

$$-\int_{-\pi}^{\pi} 3x^2\frac{e^{-inx}}{-in}dx = \int_{-\pi}^{\pi} 3x^2\frac{e^{-inx}}{in}dx = 3x^2\frac{e^{-inx}}{in(-in)}\Big|_{x=-\pi}^{\pi} - \int_{-\pi}^{\pi} 6x\frac{e^{-inx}}{in(-in)}dx.$$

The first term vanishes. So, we just gotta deal with the second term which is:

$$\int_{-\pi}^{\pi} 6x\frac{e^{-inx}}{(in)^2}dx.$$

One last integration by parts shows that

$$\int_{-\pi}^{\pi} 6x\frac{e^{-inx}}{(in)^2}dx = 6x\frac{e^{-inx}}{(in)^2(-in)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 6\frac{e^{-inx}}{(in)^2(-in)}dx.$$

By periodicity, the last term vanishes. So in total, the Fourier coefficient shall be

$$\frac{1}{2\pi}\left(\left(x^3\frac{e^{-inx}}{-in}\right)_{x=-\pi}^{\pi} + 6x\frac{e^{-inx}}{(in)^2(-in)}\Big|_{-\pi}^{\pi}\right)$$

$$= \frac{1}{2\pi}\left(\frac{\pi^3(-1)^n}{-in} - \frac{-\pi^3(-1)^n}{-in} + \frac{6\pi(-1)^n}{-(in)^3} - \frac{6(-\pi)(-1)^n}{-(in)^3}\right).$$

$$= \frac{1}{2\pi}\left(\frac{2\pi^3(-1)^n}{-in} + \frac{12\pi(-1)^n}{-(i^3n^3)}\right)$$

$$= \frac{i\pi^2(-1)^n}{n} - \frac{i6(-1)^n}{n^3}.$$

At the point 8π we use periodicity. Our function is 2π periodic, by its very definition. Consequently, the value at $8\pi = 4(2\pi) + 0$ is the same as the value at 0 which is 0.

4. Hitta polynomet, p, av högst grad två som minimera

$$\int_{-2}^{2} |\sinh(2x) - p(x)|^2 dx.$$

(10 p)

You can either build up the orthogonal polynomials on the interval by hand, or you can use the French polynomials. It's up to you.

The Legendre polynomials are orthogonal on $L^2[-1,1]$. Let P_n denote the n^{th} Legendre polynomial. Then we compute

$$\int_{-2}^{2} P_n(x/2)P_m(x/2)dx = 2\int_{-1}^{1} P_n(t)P_m(t)dt = \begin{cases} 0 & n \neq m \\ \frac{4}{2n+1} & n = m \end{cases}$$

where we have used the change of variables $t = x/2$, so $2dt = dx$. Thus we see that the polynomials $\{P_n(x/2)\}_{n\geq 1}$ are orthogonal on $L^2[-2,2]$. We use these to expand our function. The theory dictates that the coefficients are

$$c_n = \frac{\int_{-2}^{2} P_n(x/2)\sinh(2x)dx}{\frac{4}{2n+1}},$$

and the polynomial we seek is

$$\sum_{n=0}^{2} c_n P_n(x/2).$$

5. Lös problemet:

$$\begin{aligned} u_t - u_{xx} &= \cosh(x), \quad 0 < x < 4, \quad t > 0 \\ u(x,0) &= v(x), \\ u(0,t) &= 0, \\ u(4,t) &= 0. \end{aligned}$$

(10 p)

Woop woop, the inhomogeneity in the PDE is *time independent!* This means we can deal with it using a steady-state solution. So, we seek $f(x)$ to solve

$$-f''(x) = \cosh(x).$$

It just so happens that the derivative of cosh is sinh, and the derivative of sinh is cosh. No minus signs (a small advantage versus sines and cosines). Thus a solution to our equation is given by:

$$f(x) = -\cosh(x) + ax + b.$$

The other stuff, the a and the b come from the solution to the homogeneous ODE, $f''(x) = 0$. We'd rather not mess up the boundary condition, so let us figure out good values of a and b so that

$$f(0) = 0 = f(4).$$

For the first condition, we have

$$-\cosh(0) + b = -1 + b = 0 \implies b = 1.$$

For the second condition we have

$$-\cosh(4) + 4a + 1 = 0 \implies a = \frac{\cosh(4) - 1}{4}.$$

So,

$$f(x) = -\cosh(x) + \frac{\cosh(4) - 1}{4}x + 1.$$

Next, we solve the homogeneous PDE. OBS! We gotta modify our initial condition, cause when we add the steady state solution, if we don't modify the IC, then the steady state solution part will screw it up. So, we solve the problem:

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 < x < 4, \quad t > 0 \\ u(x, 0) &= v(x) - f(x), \\ u(0, t) &= 0, \\ u(4, t) &= 0. \end{aligned}$$

Our full solution will then be equal to

$$u(x, t) + f(x).$$

To solve the homogeneous ODE, we can use separation of variables! Write (remember, means to an end) $u = XT$, and stick in the PDE:

$$T'X - X''T = 0 \implies \frac{T'}{T} = \frac{X''}{X} = \text{constant}.$$

Since we got more information on the X variable, let us begin there. We use the BCs:

$$X'' = \text{constant} * X, \quad X(0) = X(4) = 0.$$

In general, solutions will be linear combinations of $e^{x\sqrt{\text{constant}}}$. I leave it to you to verify that if the constant is positive, the only solution is $X = 0$. Not interesting nor useful. If the constant is negative, then it is equivalent to use sine and cosine. The only non-zero solutions are constant multiples of

$$\sin(n\pi x/4), \quad n \in \mathbb{N}.$$

So, we have found

$$X_n(x) = \sin(n\pi x/4), \quad n \in \mathbb{N}, \quad \text{with constant } -\frac{n^2\pi^2}{16}.$$

The equation for the partner function, T_n is then

$$\frac{T_n'}{T_n} = -\frac{n^2\pi^2}{16}.$$

Up to constant multiples, the solution is

$$T_n(t) = e^{-t\frac{n^2\pi^2}{16}}.$$

Now, since the PDE is homogeneous, we may use the principle of superposition (i.e. smashing everything together in a series) to write

$$u(x,t) = \sum_{n\geq 1} c_n X_n(x) T_n(t).$$

To get the constant factors, we use the IC which since $T_n(0) = 0$ for all n says

$$u(x,0) = \sum_{n\geq 1} c_n X_n(x) = v(x) - f(x).$$

Hence, the coefficients are the Fourier coefficients with respect to the functions X_n. (Sturm-Liouville theory magically gives us the fact that these functions are an orthogonal basis for $L^2(0,4)$, so we can expand any function in terms of these X_n).

$$c_n = \frac{\int_0^4 X_n(x)(v(x) - f(x))dx}{\int_0^4 X_n(x)^2 dx}.$$

6. Lös problemet:

$$u_t - u_{xx} = G(x,t), \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x,0) = v(x).$$

(10 p)

You're welcome. This is identical to a problem on THE LAST TWO EXAMS! So, I REALLY hope y'all managed to get it right this time!!

7. Lös problemet:

$$\begin{cases} u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 & 0 \le r \le 1, |\theta| \le \pi \\ u(1,\theta) = \sin^2\theta + \cos\theta \end{cases}$$

(10 p)

Stay calm and carry on. I know things look a little scary in polar coordinates. At least, you can basically follow your nose here. Write $u = R(r)\Theta(\theta)$, and let's see if we can solve this problem. We put this into the PDE,

$$R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' = 0.$$

Now let's divide by $R\Theta$,

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = 0.$$

We move Θ stuff to the right side and multiply everything by r^2:

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}.$$

Woop woop, both sides got to be constant! The Θ side looks WAY easier, so let's deal with it first. We need to find Θ so that

$$-\Theta'' = \text{constant} * \Theta.$$

What other information do we have to go with? Well, remember, this problem is in a disk! So, the function Θ sure as heck better be 2π periodic! As in the previous problem we had, the only way to satisfy both the equation AND be 2π periodic is for (up to constant multiples)

$$\Theta = \Theta_m = e^{im\theta}, \quad m \in \mathbb{Z}, \text{ constant} = m^2.$$

(Equivalently, we could write Θ as a linear combination of $\sin(m\theta)$ and $\cos(m\theta)$, but I find the above way more simple).

Now, we use the value of the constant to find the partner function, $R_m(r)$. The equation for this guy is:

$$r^2 \frac{R_m''}{R_m} + r\frac{R_m'}{R_m} = m^2.$$

Let us multiply everything by R_m to get rid of those pesky fractions:

$$r^2 R_m'' + rR_m' = m^2 R_m.$$

Now we can subtract the right side to get a nice homogeneous ODE:

$$r^2 R_m'' + rR_m' - m^2 R_m = 0.$$

This ODE even has a name! It's an Euler equation. Solutions will be of the form $R_m(x) = x^a$ for some a. Plugging into the equation:

$$r^2 a(a-1)r^{a-2} + rar^{a-1} - m^2 r^a = 0,$$

in other words

$$a(a-1)r^a + ar^a - m^2 r^a = 0 \implies (a^2 - a + a - m^2) = 0 \implies a = \pm m.$$

So, our solution will look like a linear combination of

$$r^m \text{ and } r^{-m}.$$

OBS! If $m = 0$ then these two are the same. They no longer form a basis. So let us investigate that case in further detail. When $m = 0$ the equation is

$$r^2 R_0'' + r R_0' = 0 \implies \frac{R_0''}{R_0'} = (\log(R_0'))' = -\frac{r}{r^2} = -\frac{1}{r}.$$

Hence in this case

$$\log(R_0') = -\log(r) + \text{ constant }.$$

So,

$$R_0' = \frac{1}{r} * \text{ constant }.$$

We therefore obtain

$$R_0 = A \log(r) + B, \text{ for some constants } A \text{ and } B.$$

Now, the function $\log(r)$ is not very well behaved at $r = 0$. So we do not use this part. Note that for $m = 0$ our function $\Theta_m = 1$. So, the $m = 0$ case just yields a constant term. Now let's continue with the non-zero m. Note that for $\pm m$, $(\pm m)^2$ is the same. So we can just consider $m > 0$.

Then, if $m > 0$, the term r^{-m} is not very nicely behaved at $r = 0$ which lies smack in the middle of where we're solving our problem. So we shall also cast away those ill-behaved solutions. Thus, up to constant multiples, our solutions look like

$$R_m(r) = r^{|m|}, \quad \Theta_m(\theta) = e^{im\theta}.$$

Now, let us use the homogeneity of the PDE to smash them all together, writing

$$u(r, \theta) = \sum_{m \in \mathbb{Z}} c_m r^{|m|} e^{im\theta}.$$

When $r = 1$, we have boundary condition

$$u(1, \theta) = \sum_{m \in \mathbb{Z}} c_m e^{im\theta} = \sin^2 \theta + \cos \theta.$$

So, we recognize the left side as a Fourier series, and the right side as a 2π periodic function, hence

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sin^2\theta + \cos\theta) e^{-im\theta} d\theta.$$

You don't actually have to compute these integrals.

8. Beräkna för $x \in \mathbb{R}$

$$\sum_{n=-\infty}^{\infty} |J_n(x)|^2.$$

Tips: funktionen $e^{ix\sin(t)}$ är 2π periodisk in t-variabeln och

$$e^{ix\sin(t)} = \sum_{n=-\infty}^{\infty} J_n(x) e^{int}.$$

There's always got to be a wild-card problem. Something for those who bore easily. At the same time, hopefully the hint was helpful... I also intentionally paired this problem with the first two theory problems... You see, using the expansion of the function in the hint,

$$\int_{-\pi}^{\pi} |e^{ix\sin(t)}|^2 dt = \int_{-\pi}^{\pi} (\sum_{n\in\mathbb{Z}} J_n(x) e^{int}) \overline{\sum_{m\in\mathbb{Z}} J_m(x) e^{imt}} dt$$

$$= \sum_{m,n\in\mathbb{Z}} \int_{-\pi}^{\pi} J_n(x) \overline{J_m(x)} e^{int} \overline{e^{imt}} dt = 2\pi \sum_{n\in\mathbb{Z}} |J_n(x)|^2.$$

Here we have used the fact that the stuff with $J_n(x)$ and $J_m(x)$ is independent of t, so we only need to think about the integrals:

$$\int_{-\pi}^{\pi} e^{int} \overline{e^{imt}} dt = \begin{cases} 2\pi & m = n \\ 0 & m \neq n. \end{cases}$$

So, only the terms with $m = n$ survive! These also pick up a factor of 2π. On the other hand, for all real x and t,

$$|e^{ix\sin(t)}| = 1.$$

Thus

$$\int_{-\pi}^{\pi} |e^{ix\sin(t)}|^2 dt = \int_{-\pi}^{\pi} 1 dt = 2\pi = 2\pi \sum_{n\in\mathbb{Z}} |J_n(x)|^2.$$

So, the sum is simply one. UNO!

Lycka till! May the force be with you! ♡ Julie Rowlett.