Lösningar till Fourieranalys MVE030 och Fourier Metoder MVE290 5.juni.2018

Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. Låt f vara en 2π-periodisk funktion med $f \in \mathcal{C}^1(\mathbb{R})$. Fourierkoefficienterna av f,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

och Fourierkoefficienterna av f'

$$c_n' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx.$$

Bevisa att Fourierkoefficienterna c_n av f och Fourierkoefficienterna c_n' av f' uppfyller

$$c_n' = inc_n.$$

It's in the theory proof document already!

2. Låt $g \in L^1(\mathbb{R})$ med

$$\int_{\mathbb{R}} g(x) dx = 1.$$

Antar att f är kontinuerlig och begränsad. Låt

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}, \quad \epsilon > 0.$$

Bevisa:

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = f(x) \quad \forall x \in \mathbb{R}.$$

It's in the theory proof document already! Note that this is a slightly more simple version because:

- We assume f is continuous so its left and right hand limits are always the same.
- We assume f is bounded so there is one case rather than two cases (the other case doesn't assume f is bounded, but instead assumes that g vanishes outside some bounded interval).

So if you can do the proof in the theory list, then you can do the proof above, and it should actually be quicker and easier!!

3. Beräkna:

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{1+n^2}.$$

Tips: Beräkna den komplexa Fourierserien till den 2π-periodiska funktion $f(x)$ som är lika med $\cosh(x)$ i $(-\pi,\pi)$. Vad är seriens summa i punkten 2π?

All right, Fourier series of $\cosh(x)$. I am lazy so going to recycle some older solutions: Let us compute the Fourier coefficients of e^{-x}:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}dx = \frac{1}{-2\pi(1+in)}\left(e^{-x(1+in)}\right)_{x=-\pi}^{\pi}$$

$$= \frac{-1}{2\pi(1+in)}\left(e^{-\pi(1+in)} - e^{\pi(1+in)}\right) = \frac{1}{2\pi(1+in)}\left(e^{\pi}e^{i\pi n} - e^{-\pi}e^{-i\pi n}\right)$$

$$= \frac{1}{\pi(1+in)}(-1)^n\sinh(\pi).$$

Now we do the same for e^x:

$$\int_{-\pi}^{\pi} e^{x}e^{-inx}dx = \left.\frac{e^{x(1-in)}}{1-in}\right|_{x=-\pi}^{x=\pi} = \frac{e^{\pi}e^{-in\pi}}{1-in} - \frac{e^{-\pi}e^{in\pi}}{1-in} = (-1)^n\frac{2\sinh(\pi)}{1-in}.$$

So dividing by 2π we have

$$\frac{1}{\pi}(-1)^n\frac{\sinh(\pi)}{1-in}.$$

Since

$$\cosh(x) = \frac{e^x+e^{-x}}{2},$$

the Fourier coefficients of $\cosh(x)$ are

$$c_n = \frac{1}{2}\left(\frac{1}{\pi}(-1)^n\frac{\sinh(\pi)}{1-in} + \frac{1}{\pi(1+in)}(-1)^n\sinh(\pi)\right).$$

We could simplify this up if we want to, but it's not really necessary. Just to make it pretty though,

$$c_n = \frac{1}{2}\left(\frac{(-1)^n\sinh(\pi)}{\pi(1-in)} + \frac{(-1)^n\sinh(\pi)}{\pi(1+in)}\right) = \frac{(-1)^n\sinh(\pi)}{2\pi}\left(\frac{1+in+1-in}{(1+in)(1-in)}\right)$$

$$= \frac{(-1)^n \sinh(\pi)}{\pi(1+n^2)}.$$

The Fourier series is

$$\sum_{n\in\mathbb{Z}} c_n e^{inx} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n \sinh(\pi)}{\pi(1+n^2)} e^{inx}.$$

At the point 2π, we recall the fact that the function is 2π PERIODIC. So, the value at $2\pi = 0 + 2\pi$ is the same at the value at 0, and $\cosh(0) = 1$. We thereby obtain the interesting identity:

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{1+n^2} = \frac{\pi}{\sinh(\pi)}.$$

4. Hitta polynomet, p, av högst grad två som minimera

$$\int_{-2}^{2} |\sinh(x) - p(x)|^2 dx.$$

We seek the aid of the French polynomials, who are surely basking in the sun someplace. Plenty of time for sunbathing later, come over here and help us solve this problem, *s'il vous plait!* There is no weight function, so we ought to use the Legendre polynomials. The Legendre polynomials $P_n(t)$ are pairwise orthogonal if t goes from -1 to 1. So, if x is from -2 to 2, then we define

$$t := x/2.$$

Then we compute

$$\int_{-2}^{2} P_n(x/2)P_m(x/2)dx = \int_{-1}^{1} P_n(t)P_m(t)(2dt) = \begin{cases} 0 & n \neq m \\ \frac{4}{n+1} & n = m \end{cases}$$

This calculation is also found in β-12.2. It shows us that the *modified Legendre polynomials,* defined to be $P_n(x/2)$ rather than $P_n(x)$ are orthogonal on $[-2, 2]$. Hence, we can expand our function $\sinh(x)$ in terms of these polynomials, because they are an orthogonal basis for the Hilbert space, $\mathcal{L}^2([-2, 2])$. The function $\sinh(x)$ is an element of this Hilbert space. If we were to expand in a full series:

$$\sum_{n\geq 0} a_n P_n(x/2), \quad a_n = \frac{\langle \sinh(x), P_n(x/2)\rangle}{||P_n(x/2)||^2}.$$

Here is where it's important to know your scalar product in the Hilbert space:

$$\langle \sinh(x), P_n(x/2) \rangle = \int_{-2}^{2} \sinh(x) \overline{P_n(x/2)} dx.$$

Since P_n is real, the complex conjugation doesn't do anything. It is also important to know the norm squared,

$$||P_n(x/2)||^2 = \int_{-2}^{2} |P_n(x/2)|^2 dx = \frac{4}{2n+1},$$

cause we computed this integral with the help of β above. Now, we are not asked for the full series (phew!) just the first three terms, because the best approximation theorem says that the best approximation with just polynomial of up to degree 2 is the first three terms of this Fourier-Legendre expansion. So, the coefficients we seek are:

$$a_n = \frac{\int_{-2}^{2} \sinh(x) P_n(x/2) dx}{\frac{4}{2n+1}}, \quad n = 0, 1, 2,$$

and the polynomial we seek is:

$$\sum_{n=0}^{2} a_n P_n(x/2).$$

5. Lös problemet:

$$\begin{aligned} u_t - u_{xx} &= e^{x+t}, \quad 0 < x < 4, \quad t > 0 \\ u(x,0) &= v(x), \\ u_x(0,t) &= 0, \\ u_x(4,t) &= 0. \end{aligned}$$

We shall seek a series solution as done in previous exams. First consider the x part. In the homogeneous case, separating variables we would solve for X to satisfy

$$X'' = \lambda X, \quad X'(0) = X'(4) = 0.$$

The general solutions are exponentials, but only the complex exponentials (corresponding to trig functions, equivalently) yield non-zero solutions. These non-zero solutions are multiples of

$$X_n(x) = \cos(n\pi x/4), \tag{1}$$

but we shall deal with the constant stuff later. So, we write a series

$$\sum_{n\geq 0} c_n(t)X_n(x),$$

and we plug the series into the PDE:

$$\sum_{n\geq 0} c_n'(t)X_n - a_n(t)X_n''(x) = e^{x+t}.$$

Next we use the fact that

$$X_n''(x) = -\frac{n^2\pi^2}{16}X_n(x).$$

So, our series becomes

$$\sum_{n\geq 0} c_n'(t)X_n(x) + c_n(t)\frac{n^2\pi^2}{16}X_n(x) = e^{x+t}.$$

Consolidate in the sum:

$$\sum_{n\geq 0} X_n(x)\left(c_n'(t) + c_n(t)\frac{n^2\pi^2}{16}\right) = e^{x+t}.$$

Now, let us look at the right side. $e^{x+t} = e^x e^t$. We need to have the right side as a series involving X_n as well. We can do this by expanding e^x in terms of the basis X_n, so we define

$$a_n = \frac{\int_0^4 e^x X_n(x)dx}{\int_0^4 X_n^2(x)dx}. \tag{2}$$

This should look awfully familiar to the best approximation problem, because it is the same concept. We therefore expand the function e^x as

$$e^x = \sum_{n\geq 0} a_n X_n(x).$$

Now our equation becomes

$$\sum_{n\geq 0} X_n(x)\left(c_n'(t) + c_n(t)\frac{n^2\pi^2}{16}\right) = e^t\sum_{n\geq 0} a_n X_n(x) = \sum_{n\geq 0} e^t a_n X_n(x).$$

We can now equate coefficients of X_n on the left and the right:

$$c_n'(t) + \frac{n^2\pi^2}{16} c_n(t) = e^t a_n.$$

This is an ODE which can be found in β 9.1.3. For notational convenience, let us write

$$\lambda_n = \frac{n^2\pi^2}{16}. \tag{3}$$

Then our solution is

$$c_n(t) = \frac{a_n e^t + b_n e^{-\lambda_n t}}{\lambda_n + 1}, \qquad b_n \text{ will be determined below.} \tag{4}$$

Note that $\lambda_n \geq 0$ for all n, so we are NOT dividing by zero. Phew! We will need to determine the as of now unknown numbers b_n using the initial condition. Let us write our solution now as

$$u(x,t) = \sum_{n \geq 0} c_n(t) X_n(x). \tag{5}$$

We need

$$u(x,0) = \sum_{n\geq 0} c_n(0) X_n(x) = \sum_{n \geq 0} \frac{a_n + b_n}{\lambda_n + 1} X_n(x) = v(x).$$

This shows that the numbers in front of the $X_n(x)$ need to be the Fourier coefficients of the function $v(x)$ with respect to the basis $\{X_n\}$. Thus we need

$$\frac{a_n + b_n}{\lambda_n + 1} = \frac{\int_0^4 v(x) X_n(x) dx}{\int_0^4 X_n^2(x) dx},$$

and hence the coefficients

$$b_n = (\lambda_n + 1) \frac{\int_0^4 v(x) X_n(x) dx}{\int_0^4 X_n^2(x) dx} - a_n.$$

Our full solution is (5) with a_n defined in (2), b_n defined above, λ_n defined in (3), $c_n(t)$ defined in (4), and $X_n(x)$ defined in (1).

6. Lös problemet:

$$u_t - u_{xx} = G(x,t), \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x,0) = v(x).$$

Okay, so this one is a bit of a Midsommarklapp. (Too early for a Julklapp). It's the same as the previous exam. Hope y'all studied that! I copy the solution here: We have an *inhomogeneous* heat equation which depends on both time and space. Not a problem. We hit the PDE with the Fourier transform on $x \in \mathbb{R}$ variable:

$$\hat{u}_t(\xi,t) - \widehat{u_{xx}}(\xi,t) = \hat{G}(\xi,t).$$

We use β 13.2.F10 with $n = 2$ there:

$$\widehat{u_{xx}}(\xi,t) = (i\xi)^2 \hat{u}(\xi,t).$$

So the equation is

$$\hat{u}_t(\xi,t) + \xi^2 \hat{u}(\xi,t) = \hat{G}(\xi,t).$$

Stay calm. This is just an ODE for u with respect to the variable t. We look it up in β. We find the solution is given in β 9.1.3. First, we compute

$$\exp(-\int \xi^2 dt) = e^{-\xi^2 t} \text{ don't need integration constant here according to } \beta.$$

Next, we compute the solution is

$$\hat{u}(\xi,t) = e^{-\xi^2 t} \left(\int_0^t e^{\xi^2 s} \hat{G}(\xi,s) ds + C \right).$$

We use the IC to determine C:

$$\hat{u}(\xi,0) = \hat{v}(\xi) = C,$$

so

$$\hat{u}(\xi,t) = e^{-\xi^2 t} \left(\int_0^t e^{\xi^2 s} \hat{G}(\xi,s) ds + \hat{v}(\xi) \right) = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi,s) ds + e^{-\xi^2 t} \hat{v}(\xi).$$

We know (or look it up in β) that to get a product from the Fourier transformation, we start with a convolution. In the second term, we can look up already that:

$$e^{-x^2/(4t)} (4\pi t)^{-1/2} \text{ Fourier transforms to } e^{-\xi^2 t}.$$

We get this from β 13.2 F37. Well, then similarly, the same formula shows that

$$e^{-x^2/((4(t-s))}(4\pi(t-s))^{-1/2} \text{ Fourier transforms to } e^{-\xi^2(t-s)}.$$

So, our solution is given by the sum of the convolutions:

$$u(x,t) = \int_{\mathbb{R}} \int_0^t e^{-(x-y)^2/(4(t-s))}(4\pi(t-s))^{-1/2} G(y,s) ds dy + \int_{\mathbb{R}} e^{-(x-y)^2/(4t)}(4\pi t)^{-1/2} v(y) dy.$$

7. Lös problemet i annulusen:

$$\begin{cases} u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 & 1 < r < 2, |\theta| \leq \pi \\ u(1,\theta) = 0 & |\theta| \leq \pi \\ u(2,\theta) = 1 - \frac{\theta^2}{\pi^2} & |\theta| \leq \pi. \end{cases}$$

Stay calm and carry on. Did I fool you into thinking Bessel functions would come out of this? Sorry, that was kind of my thinking... Need to keep you on your toes after all! We take the PDE and do our favorite thing: separate variables. Write

$$u = R(r)\Theta(\theta),$$

plug into the PDE (remember, u is not going to be like this in the end, this is just a means to an end):

$$R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' = 0.$$

Move $r^{-2}R\Theta''$ to the right side, divide both sides by $R\Theta$:

$$\frac{R''}{R} + \frac{R'}{rR} = -r^{-2}\frac{\Theta''}{\Theta}.$$

Multiply both sides by r^2:

$$r^2\left(\frac{R''}{R} + \frac{R'}{rR}\right) = -\frac{\Theta''}{\Theta}.$$

Each side depends on a different variable so both sides are constant. Work with the simple side first,

$$-\frac{\Theta''}{\Theta} = \lambda \implies -\Theta'' = \lambda\Theta.$$

What do we know about Θ? What's the *geometry* of the problem? We are working in an *annulus.* This means that the function must be periodic in the θ variable, because $\theta = \pi$ is the same point as $\theta = 3\pi$ and $\theta = 5\pi$, etc. The function Θ is 2π periodic. So, we want to solve:

$$-\Theta'' = \lambda\Theta, \quad \Theta(\theta + 2\pi) = \Theta(\theta).$$

In general the solutions will be exponential functions, and with the periodicity consideration, we compute that the solutions are

$$\Theta_n = e^{in\theta}, \quad -\Theta_n'' = -(in)^2\Theta_n = n^2\Theta_n \implies \lambda_n = n^2.$$

Now we use this information to solve for the partner function, R_n. The equation for R_n is

$$r^2\left(\frac{R''}{R} + \frac{R'}{rR}\right) = -\frac{\Theta''}{\Theta} = \lambda = \lambda_n = n^2.$$

Re-arranging:

$$r^2R'' + rR' = n^2R \implies r^2R'' + rR' - n^2R = 0.$$

We ought to think about two cases: $n = 0$ and $n \neq 0$. In case $n = 0$ the equation is

$$r^2R'' + rR' = 0.$$

This is a first order ODE in R'. We can divide through by r and obtain

$$r(R')' + R' = 0 \iff r(R')' = -R' \iff \frac{(R')'}{R'} = -\frac{1}{r}.$$

The left side is the derivative of $\ln(R')$, so we have

$$\ln(R')' = -\frac{1}{r}.$$

We can integrate both sides:

$$\ln(R') = -\ln(r) + C.$$

Hence,

$$R' = e^{-\ln(r)+C} = \frac{e^C}{r}.$$

Again we integrate both sides:

$$R(r) = e^C\ln(r) + B.$$

We use the boundary condition at $r = 1$ to compute that $B = 0$. Let us also re-name $e^C = a_0$. So, the solution for $n = 0$ is

$$R_0(r) = a_0 \ln(r).$$

For $n \neq 0$ the equation for R is an Euler equation. The solution is a function of the form $R(r) = r^x$. Plug such a function into the equation:

$$r^2(x)(x-1)r^{x-2}+r(x)r^{x-1}-n^2r^x = 0 \iff x(x-1)+x-n^2 = 0 \iff x^2 = n^2.$$

So, we have two solutions, r^n and r^{-n}. The general solution looks like

$$a_n r^n + b_n r^{-n}.$$

What should the coefficients be? We use the boundary conditions. When $r = 1$ the solution is supposed to be zero. So, we want

$$a_n + b_n = 0 \implies b_n = -a_n.$$

Since the PDE is homogeneous, we can smash all our solutions together into a series:

$$\sum_{n\in\mathbb{Z}} e^{in\theta} a_n \left(r^n - r^{-n}\right). \tag{6}$$

When $r = 2$, we want this to be equal to $1 - \frac{\theta^2}{\pi^2}$, so we write

$$\sum_{n\in\mathbb{Z}} e^{in\theta} a_n \left(2^n - 2^{-n}\right) = 1 - \frac{\theta^2}{\pi^2}.$$

The left side looks awfully much like a Fourier series... Let us make it the Fourier series of the right side. We would need

$$a_n \left(2^n - 2^{-n}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{\theta^2}{\pi^2}\right) e^{-in\theta} d\theta.$$

Consequently,

$$a_n = \frac{1}{2\pi(2^n - 2^{-n})} \int_{-\pi}^{\pi} \left(1 - \frac{\theta^2}{\pi^2}\right) e^{-in\theta} d\theta, \quad n \in \mathbb{Z} \setminus \{0\}. \tag{7}$$

We next need to compute the coefficient a_0. To do this, we need to compute the 0^{th} Fourier coefficient for the function $1 - \frac{\theta^2}{\pi^2}$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 - \frac{\theta^2}{\pi^2} d\theta = \frac{1}{2\pi} \left(2\pi - \frac{2\pi^3}{3\pi^2}\right) = 1 - \frac{1}{3} = \frac{2}{3}.$$

So, we need the coefficient

$$a_0 \ln(2) = \frac{2}{3} \implies a_0 = \frac{2}{3\ln(2)}.$$

Hence our full solution is

$$u(r,\theta) = \frac{2}{3\ln(2)} \ln(r) + \sum_{n \in Z \setminus \{0\}} e^{in\theta} a_n \left(r^n + r^{-n}\right),$$

with a_n given by equation (7) for $n \neq 0$.

8. Om $f(x)$ har Fouriertransformen $\hat{f}(\xi)$ vad är Fouriertransformen av $\cos(x) f(x/2)$?

This problem is all about the properties of the Fourier transform, also relatively straightforward, just keep calm and carry on. Let's just write out the definition:

$$\int_{\mathbb{R}} e^{-ix\xi} \cos(x) f(x/2) dx.$$

Use the complex representation of cosine:

$$\int_{\mathbb{R}} \frac{1}{2} \left(e^{ix} + e^{-ix}\right) e^{-ix\xi} f(x/2) dx.$$

Deal with each term separately. First write

$$\clubsuit = \frac{1}{2} \int_{\mathbb{R}} e^{ix} e^{-ix\xi} f(x/2) dx = \frac{1}{2} \int_{\mathbb{R}} e^{-ix(\xi-1)} f(x/2) dx.$$

This is looking pretty close to a Fourier transform. Just need to change out variables. Let $y = x/2$. Then $dy = dx/2$ so $2dy = dx$. Also, $x = 2y$. So, our expression becomes

$$\clubsuit = \int_{\mathbb{R}} e^{-iy(2\xi-2)} f(y) dy = \hat{f}(2\xi - 2).$$

Next, we consider the second term, letting

$$\diamondsuit = \frac{1}{2} \int_{\mathbb{R}} e^{-ix} e^{-ix\xi} f(x/2) dx.$$

We will proceed similarly: combine the exponentials and change variables:

$$\diamondsuit = \frac{1}{2} \int_{\mathbb{R}} e^{-ix(\xi+1)} f(x/2) dx = \int_{\mathbb{R}} e^{-iy(2\xi+2)} f(y) dy = \hat{f}(2\xi + 2).$$

The total Fourier transform is thus

$$\clubsuit + \diamondsuit = \hat{f}(2\xi - 2) + \hat{f}(2\xi + 2).$$

Lycka till! May the force be with you! ♡ Julie Rowlett.