FOURIER ANALYSIS & METHODS

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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According to Gerry, Fourier Analysis is "A collection of related techniques for solving the most important partial differential equations of physics (and chemistry)." For example, we're going to be solving partial differential equations, abbreviated PDEs

- Δ Laplace equations (related to computing energy of quantum particles)
- $\Box$ wave equations (describes the propagation of waves, hence also of light and electromagnetic waves)
- heat equation (describes the propagation of heat, is the quintessential diffusion equation)

A general feature of PDEs is that they are HARD. There is no beautiful unifying theory which can guide us to the solution of all PDEs. For this reason, we must study a variety of methods to have a decent chance at solving PDEs. These methods come in many different mathematical flavors. Hopefully you will like some of them.

Well, we have been a bit impolite here, talking about PDEs without properly introducing them. Let us remedy this.

Definition 1. A PDE is an equation for an unknown function (unsub) which depends on two or more independent variables. Writing u for the unknown function,

$$u : \mathbb{R}^n \to \mathbb{R}$$

or

$$u : \mathbb{R}^n \to \mathbb{C}.$$

The PDE contains u and may contain its partial derivatives. It may also contain other, *specified* (that is *not* unknown but rather, specified) functions.

1.1. The first method: Separation of variables (SV).

If you come to the (obligatory for Kf, option for TM and F) extra three lectures, you will learn how to classify every PDE on the planet. For the great majority of these, we have no hope to solve then analytically (that is, to write down a mathematical formula as the solution to the PDE).[1] However, for a special few, we can solve them. The first tricky way to solve a PDE, that is an equation for an unknown function which

[1]Probabilistically speaking, we can *almost sure* not solve PDEs analytically.

depends on several variables, all jumbled up together, is to separate those jumbled up variables. If we can do that, then we have a ray of hope of turning the PDE into one or more ODEs. ODEs are called ordinary for a reason. They're simpler than PDEs.

So, to introduce the technique of separation of variables, let's think about a really down-to-earth example. A vibrating string. (Could be on a guitar, piano, violin, or whatever instrument you prefer). The ends of the string are held fixed, so they're not moving. You know this if you play or watch people play guitar. One end is fixed at the bottom, and you slide your hand to various places on the string, to hold it down, to get the notes you want. We're going to see something rather interesting mathematically about how this all works. Let's mathematicize the string, by identifying it with the interval $[0, \ell] \subset \mathbb{R}$. The string length is ℓ. Let's define

$$u(x,t) := \text{ the height of the string at the point } x \in [0,\ell] \text{ at time } t \in [0,\infty[.$$

Then, let's just define the sitting-still height to be height 0. So, the fact that ends are sitting still means that

$$u(0,t) = u(\ell,t) = 0 \quad \forall t.$$

A positive height means above the sitting-still height, whereas a negative height means under the sitting-still height. The wave equation (I'm not going to derive it, but maybe you clever physics students can do that?) says that:

$$u_{xx} = \mathfrak{c}^2 u_{tt}.$$

The constant $\mathfrak{c}$ depends basically on how fast the string vibrates. Now, because we're in a math class, we're going to use funky time units and just assume $c = 1$. The reason we can do that is if we defined time units $\tau = \mathfrak{c}t$, then $u_{\tau\tau} = \mathfrak{c}^2 u_{tt}$. So, using the funky time units τ (which are just the old t time units, whatever they were, scaled by $\mathfrak{c}$), the wave equation becomes

$$u_{xx} = u_{\tau\tau}.$$

Mathematicians do this trick *all the time!* Engineers hate it. They complain when we say *without loss of generality assume that* $\mathfrak{c} = 1$. So, I've decided to join the engineers this time and keep $\mathfrak{c}$ in the equation.[2]

What we've got is a PDE, because we got both x derivatives and t derivatives, and u depends on both x and t. Eek. The separation of variables idea is to *assume* we can write

$$u(x,t) = f(x)g(t),$$

that is a product of two functions, each of which depends only on *one* variable. (Whether this assumption is kosher remains to be determined...) Now, assuming that u is of this form, we write the PDE

$$u_{xx} = \mathfrak{c}^2 u_{tt} \iff f''(x)g(t) = \mathfrak{c}^2 f(x) g''(t).$$

[2] In my research I work with the heat and wave equations in much more complicated settings. It is standard in the theoretical math research community to take $c = 1$. This is because the math in our settings is so damn complicated, and we understand the role of c very well. So, we just ignore it, knowing that at the end of the day we could incorporate it back into the picture if we wished. We do the same thing with the heat equation (which you shall see soon enough)...

Doesn't look much better yet, but hang on there. Divide both sides by $f(x)g(t)$. We get

$$\frac{f''}{f}(x) = \mathfrak{c}^2 \frac{g''}{g}(t).$$

Stop. Think. The left side depends only on x, whereas the right side depends only on t. Hence, they both must be constant. We've got more information on x than we do on t, because we know that the ends are still. This means that

$$f(0) = f(\ell) = 0.$$

So, the equation for just f is

$$\frac{f''}{f}(x) = \text{ constant },$$

$$f(0) = f(\ell) = 0.$$

Let's give the constant a name. Call it λ. Then write

$$f''(x) = \lambda f(x), \quad f(0) = f(\ell) = 0.$$

Well, we can solve this. There are three cases to consider:

$\lambda = 0$ This means $f''(x) = 0$. Integrating both sides once gives $f'(x) =$ constant $= m$. Integrating a second time gives $f(x) = mx + b$. Requiring $f(0) = f(\ell) = 0$, well, the first makes $b = 0$, and the second makes $m = 0$. So, the solution is $f(x) \equiv 0$. The 0 solution. The waveless wave. Not too interesting.

$\lambda > 0$ The solution here will be of the form

$$f(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

Exercise 1. *Show that it is equivalent to write the solution as* $A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$, *for two constants* A *and* B. *Determine the relationship between* A *and* B *and* a *and* b. *Show that in order to guarantee that* $f(0) = f(\ell) = 0$ *you need* $a = A = B = b = 0$. *You should do this exercise, because it I strongly suspect you can do it. Think of it as a warm-up for Folland's exercises.*

Thus, with our teamwork, (me providing hints and you doing the actual work by solving the exercise) we have gotten the 0 solution again. The waveless wave. No fun there.

$\lambda < 0$ Finally, we have solution of the form

$$a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To make $f(0) = 0$, we need $a = 0$. Uh oh... are we going to get that stupid 0 solution again? Well, let's see what we need to make $f(\ell) = 0$. For that we just need

$$b\sin(\sqrt{|\lambda|}\ell) = 0.$$

That will be true if

$$|\lambda| = \frac{k^2\pi^2}{\ell^2}, \quad k \in \mathbb{Z}.$$

Super! We still don't know what b ought to be, but at least we've found all the possible f's, up to constant factors.

Just to clarify the fact that we've now found *all* solutions, we recall here a theorem from your multivariable calculus class.

Theorem 2 (Old Multivariable Calculus Theorem)**.** *Consider the second order linear homogeneous ODE,*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

If $b = c = 0$, then a basis of solutions is given by

$$\{x, 1\},$$

so that all solutions are of the form

$$u(x) = Ax + B, \quad A, B \in \mathbb{R}.$$

If $c = 0$, then a basis of solutions is $\{e^{-b/ax}\}$ so that all real solutions are given by

$$u(x) = Ce^{-bx/a}.$$

If $c \neq 0$, then a basis of solutions is one of the following three mutually exclusive sets:

(1) $\{e^{r_1 x}, e^{r_2 x}\}$ *if* $b^2 > 4ac$ *in which case*

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(2) $\{e^{rx}, xe^{rx}\}$ *if* $b^2 = 4ac$, *in which case* $r = -\frac{b}{2a}$.

(3) $\{\sin(\Im rx)e^{\Re rx}, \cos(\Im rx)e^{\Re rx}\}$ *if* $b^2 < 4ac$, *in which case* $r = -\frac{b}{2a} + \frac{i}{2a}\sqrt{4ac - b^2}$, *and* $\Im r = \frac{\sqrt{4ac-b^2}}{2a}$.

Exercise 2. *We had the equation $f''(x) = \lambda f(x)$. Translate this equation into the language of the OMC Theorem. What are the particular values of the "local variables" a, b, and c, in the statement of the theorem? What are the three different cases? Show that when you translate our equation into the language of the theorem, you get the same results. Show that the "boundary conditions," $f(0) = f(\ell) = 0$ then specify the solutions.*

So, the solutions we've found are, up to constant factors:

$$f_k(x) = \sin\left(\frac{k\pi x}{\ell}\right), \quad \lambda_k = -\frac{k^2\pi^2}{\ell^2}.$$

Let us pause, and remember an important moral value.

Patience is a virtue.

It is often a good idea to hold off on determining the constant factors in front of the sines until the *end.* This is because the less baggage we are carrying around, (i.e. the fewer symbols we got to write), the less likely we are to screw something up. So, we should remember the patience principle and be patient, wait to get the constants later.

Now, let's find the friends of f, the time functions, g which depend only on time. When we've got f_k, then

$$\frac{f_k''}{f_k} = \lambda_k = -\frac{k^2\pi^2}{\ell^2} = \mathfrak{c}^2 \frac{g_k''}{g_k}.$$

This is almost the same equation we had before. Here we have, re-arranging:

$$g_k'' = -\frac{k^2\pi^2}{\mathfrak{c}^2\ell^2} g_k.$$

Exercise 3. *Use the OMC Theorem to show that a basis of solutions is given by*

$$\left\{ \cos\left(\frac{k\pi t}{\mathfrak{c}\ell}\right), \sin\left(\frac{k\pi t}{\mathfrak{c}\ell}\right) \right\}.$$

Let us pause to think about what this means. The physics students may recognize that the numbers

$$\{|\lambda_k|\}_{k\geq 1}$$

are the resonant frequencies of the string. Basically, they determine how it sounds. The number $|\lambda_1|$ is the fundamental tone of the string. The higher $|\lambda_k|$ for $k \geq 2$ are harmonics. It is interesting to note that they are all square-integer multiples of λ_1. Here's a question: if you can "hear" the value of $|\lambda_1|$, then can you tell me how long the string is? Well, yes, cause

$$|\lambda_1| = \frac{1}{\ell^2}, \implies \ell = \frac{1}{\sqrt{|\lambda_1|}}.$$

So, you can hear the length of a string. A couple of famous unsolved math problems: can one hear the shape of a convex drum? Can one hear the shape of a smoothly bounded drum? We can talk about these problems if you're interested.

So, now that we've got all these solutions, what should we do with them? Good question...

1.2. **Superposition principle and linearity.** Superposition basically means adding up a bunch of solutions. You can think of it like adding up a bunch of solutions to get a super solution!

Definition 3. A second order linear PDE for an unknown function u of n variables is an equation for u and its mixed partial derivatives up to order two of the form

$$L(u) = f,$$

where f is a given function, and there are known functions $a(x)$, $b_i(x)$, $c_{ij}(x)$ for $x \in \mathbb{R}^n$ such that

$$L(u) = a(x)u(x) + \sum_{i=1}^n b_i(x)u_{x_i}(x) + \sum_{i,j=1}^n c_{ij}(x)u_{ij}(x).$$

In this context, L is called a *second order linear partial differential operator.*

The reason it's called linear is because it's well, linear.

Exercise 4. *For two functions u and v, which depend on n variables, show that*

$$L(u+v) = L(u) + L(v).$$

Moreover, for any constant $c \in \mathbb{R}$, show that

$$L(cu) = \mathfrak{c}L(u).$$

Definition 4. The wave operator, $\Box$, defined for $u(x,y)$ with $(x,y) \in \mathbb{R}^2$ is

$$\Box(u) = -u_{xx} + \mathfrak{c}^2 u_{tt}.$$

Exercise 5. *Verify that the wave operator is a second order linear partial differential operator.*

We have shown that the functions

$$u_k(x,t) = f_k(x) g_k(t)$$

satisfy

$$\Box u_k = 0 \forall k.$$

Hence, if we add them up this remains true:

$$\Box(u_1 + u_2 + u_3 + \ldots) = 0.$$

OBS![3] On the other hand, the equations

$$f_k'' = \lambda_k f_k \iff f_k'' - \lambda_k f_k = 0$$

do *not* add up.

Exercise 6. *Show this! That is show that the equations do not add up. Start by showing that:*

$$f_1'' + f_2'' - (\lambda_1 + \lambda_2)(f_1 + f_2) \neq 0.$$

Then, show that the equation satisfied by f_k is

$$L_k f_k = 0, \quad L_k(u) = u'' - \lambda_k u.$$

So, the point is that L_k is not the same operator *for the different k, because it is not the same λ_k. This exercise shows that one must take care when smashing solutions (i.e. superposing) together!*

When we look at the different $u_k(x,t)$ in the *wave equation,* it's all good, because it's always the same wave operator. Hence, we may indeed smash all our solutions together, include the (to be determined) coefficients, and write

$$u(x,t) = \sum_{k \geq 1} u_k(x,t) = \sum_{k \geq 1} \sin\left(\frac{k\pi x}{\ell}\right)\left(a_k \cos\left(\frac{k\pi t}{\mathfrak{c}\ell}\right) + b_k \sin\left(\frac{k\pi t}{\mathfrak{c}\ell}\right)\right),$$

and it satisfies

$$\Box u(x,t) = 0, \quad u(0,t) = u(\ell,t) = 0.$$

We've still got some unanswered questions:

(1) What are the constants a_k and b_k?
(2) If we can figure out what the constants are, then

$$\sum_{k \geq 1} \sin\left(\frac{k\pi x}{\ell}\right)\left(a_k \cos(k\pi t/\ell) + b_k \sin(k\pi t/\ell)\right),$$

is this mess going to converge?

[3] I love this Swedish expression. Nothing quite like it in the languages I know. Well, the closest is maybe 小心 which is also very cute.

Let's think about what happens when you play guitar. You gotta strum it or pluck it to make a sound. If you are Eddie Van Halen then you can also tap it which is super cool. So, we think of the instant when you pluck or strum the guitar, and call that time $t = 0$. Then, there is a function

$$u_0(x) = \text{ the height at the point } x \text{ on the string at time } t = 0.$$

If we just substitute rather blindly $t = 0$ into the series above, we get

$$\sum_{k \geq 1} \sin(k\pi x/\ell) a_k.$$

So, we're going to want that sum to equal $u_0(x)$. A few natural questions arise:

(1) When can we guarantee that we can find a_k to make the sum
$$\sum_{k \geq 1} \sin(k\pi x/\ell) a_k = u_0(x)?$$
(2) How do we find these a_k?
(3) What on earth do we do to get the b_k?

1.3. **A means to an end.** Finally, a note of caution. We *started* by separating variables. We then proceeded through several steps, which we summarize here:

(1) Separate variables in the wave equation, writing $u(x, t) = f(x)g(t)$.
(2) Get all the x dependent stuff on one side, and all the t dependent stuff on the other.
(3) Use the wave equation and *the boundary condition* to solve for the f function first. The assumption that the ends of the string are fixed, not moving, is called *a boundary condition.* The "place" where the action is happening is a string, identified with $[0, \ell] \subset \mathbb{R}$. That's a set. It's got a boundary. The boundary consists of the two endpoints. The boundary condition here is $f(0) = f(\ell) = 0$. This is called a homogeneous boundary condition.
(4) The boundary condition determines the values of the constant we called λ. We end up with a whole smattering of functions f_k with their corresponding λ_k.
(5) Use this information to solve for the "friend function," $g_k(t)$.
(6) Observe that $u_k(x, t) = f_k(x)g_k(t)$ also solves the wave equation. So, by superposition, we can smash them all together and create our ultimate super solution
$$u(x, t) = \sum_{k \geq 1} u_k(x, t).$$

You see, separation of variables is just a means to an end. We started by doing this, but then ended up with a super solution which is *not separated.* We still have a lot to learn. There are the mysterious coefficients, a_k and b_k. There is also the question about when can we solve the equation like this? Does it always work? Stay tuned for next time, where we turn up the heat (equation) and learn how to find the coefficients a_k and b_k to finalize our solution to the wave equation!