FOURIER ANALYSIS & METHODS

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Theorem 1 (Cute facts about SLPs)**.** *Let f and g be eigenfunctions for a regular SLP in an interval $[a,b]$ with weight function $w(x) > 0$. Let λ be the eigenvalue for f and μ the eigenvalue for g. Then:*

(1) $\lambda \in \mathbb{R}$ och $\mu \in \mathbb{R}$;
(2) If $\lambda \neq \mu$, then:

$$\int_a^b f(x)\overline{g(x)}w(x)dx = 0.$$

Proof: By definition we have $Lf + \lambda w f = 0$. Moreover, L is self-adjoint, so we have

$$\langle Lf, f\rangle = \langle f, Lf\rangle.$$

By being an eigenfunction,

$$Lf = -\lambda w f.$$

So combining these facts:

$$\langle Lf, f\rangle = \langle -\lambda w f, f\rangle = -\lambda\langle wf, f\rangle$$
$$= \langle f, Lf\rangle = \langle f, -\lambda w f\rangle = -\overline{\lambda}\langle f, wf\rangle.$$

Since w is real valued,

$$\langle wf, f\rangle = \int_a^b w(x)f(x)\overline{f(x)}dx = \int_a^b |f(x)|^2 w(x)dx,$$
$$\langle f, wf\rangle = \int_a^b f(x)\overline{w(x)f(x)}dx = \int_a^b |f(x)|^2 w(x)dx.$$

Since $w > 0$ and f is an eigenfunction,

$$\int_a^b |f(x)|^2 w(x)dx > 0.$$

So, the equation

$$-\lambda\langle wf, f\rangle = -\lambda\int_a^b |f(x)|^2 w(x)dx = -\overline{\lambda}\langle f, wf\rangle = -\overline{\lambda}\int_a^b |f(x)|^2 w(x)dx$$

implies

$$\lambda = \overline{\lambda}.$$

For the second part, we use basically the same argument based on self-adjointness:

$$\langle Lf, g \rangle = \langle f, Lg \rangle.$$

By assumption

$$\langle Lf, g \rangle = -\lambda \langle wf, g \rangle = -\lambda \int_a^b w(x) f(x) \overline{g(x)} dx.$$

Similarly,

$$\langle f, Lg \rangle = \langle f, -\mu wg \rangle = -\overline{\mu} \langle f, wg \rangle = -\mu \langle f, wg \rangle = -\mu \int_a^b f(x) \overline{g(x)} w(x) dx,$$

since $\mu \in \mathbb{R}$ and $w(x)$ is real. So we have

$$-\lambda \int_a^b w(x) f(x) \overline{g(x)} dx = -\mu \int_a^b f(x) \overline{g(x)} w(x) dx.$$

If the integral is non-zero, then it forces $\lambda = \mu$ which is false. Thus the integral must be zero.

1.1. **Applications to solving PDEs: divide and conquer.** Ideally, you want to deal with inhomogeneous parts one at a time. So, you break the problem down into pieces and try to solve the pieces: divide and conquer. Deal with each inhomogeneity one at a time. Then add them up. It is difficult to give a definitive formula that one can mindlessly use in every situation (like the quadratic formula for the solutions to $ax^2 + bx + c = 0$). The best tactic is to keep these principles and examples in mind (and at hand for reference while you are in the practicing/learning phase), and to just do lots and lots of problems. Occasionally though, especially in future "real world problems" you may come to a PDE which *has no solution.* So, if you are really struggling, consider the possibility that maybe what you're trying to do is impossible. On exams, though, this won't happen. In the real (research & applied) world though....

1.1.1. *Warm-up example.* Solve:

$$u(x,0) = \begin{cases} x + \pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

$$u(-\pi) = u(\pi) = 0$$

$$u_t(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi, \pi], \quad t > 0.$$

We begin by doing separation of variables. Write $u(x,t) = X(x)T(t)$. We get the equation

$$T'(t)X(x) - X''(x)T(t) = 0 \iff \frac{T'}{T} = \frac{X''}{X} = \lambda.$$

Since we have super nice BCs for X, we start with the X. We want to solve

$$X''(x) = \lambda X(x), \quad X(-\pi) = X(\pi) = 0.$$

First case: $\lambda = 0$. Then

$$X(x) = ax + b.$$

FIGURE 1. Just for fun, here is an old photo of my grandpa in his plane. Coincidentally, he is a first generation Swede (his parents immigrated from Sweden to the USA in the early 1900s). Note that I wrote *is,* because he's going on 98.

The BCs say

$$X(-\pi) = -a\pi + b = 0 \implies a\pi = b.$$

Next we need

$$X(\pi) = a\pi + b = 0 \implies b = -a\pi.$$

Combining these,

$$a\pi = -a\pi \implies a = 0 \implies b = 0.$$

So, no solution here because the zero solution doesn't count! Moving right along, let us try

$$\lambda > 0.$$

Then, our solution looks like real exponentials or equivalently sinh and cosh.

HINT: If you interval looks like $[0, l]$, it's probably easiest to work with sinh and cosh because $\sinh(0) = 0$ and $\cosh' = \sinh$. So this will often make things simpler. On the other hand, if you have an interval like $[a, b]$ with a and be not zero, it may be easier to work with the exponentials. So, that's why I'm choosing to do that here. Hence

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

The BCs require

$$X(-\pi) = ae^{-\sqrt{\lambda}\pi} + be^{\sqrt{\lambda}\pi} = 0.$$

Let's multiply by $e^{\sqrt{\lambda}\pi}$, to get

$$a + be^{2\sqrt{\lambda}\pi} = 0 \implies a = -be^{2\sqrt{\lambda}\pi}.$$

We check the other BCs

$$X(\pi) = ae^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0$$

substituting the value of a,

$$-be^{2\sqrt{\lambda}\pi}e^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0.$$

If $b = 0$ the whole solution is 0, so we assume this is not the case and divide by b. Multiplying by $e^{\sqrt{\lambda}\pi}$ we get

$$-e^{4\sqrt{\lambda}\pi} + 1 = 0 \iff e^{4\sqrt{\lambda}\pi} = 1 \iff 4\sqrt{\lambda}\pi = 0 \iff \lambda = 0,$$

which is a contradiction. So, no solutions lurking over here.

Thus, we consider $\lambda < 0$. Then our solution looks like

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

We need

$$X(-\pi) = a\cos(-\sqrt{|\lambda|}\pi) + b\sin(-\pi\sqrt{|\lambda|}) = 0 = a\cos(\sqrt{|\lambda|}\pi) - b\sin(\sqrt{|\lambda|}\pi),$$

where we use the evenness of cosine and oddness of sine. We also need

$$X(\pi) = a\cos(\sqrt{|\lambda|}\pi) + b\sin(\sqrt{|\lambda|}\pi) = 0.$$

Adding these equations we see that we need

$$a\cos(\sqrt{|\lambda|}\pi) = 0 \implies a = 0 \text{ or } \sqrt{|\lambda|} = \frac{(2k+1)}{2}, \quad k \in \mathbb{Z}.$$

Subtracting these equations we see that we need

$$b\cos(\sqrt{|\lambda|}\pi) = 0 \implies b = 0 \text{ or } \sqrt{|\lambda|} = \frac{2k}{2}, \quad k \in \mathbb{Z}.$$

I know it looks weird but I wrote it this way to make it looks similar to the one with the cosine. Now, the number $\sqrt{|\lambda|}$ can only have one value. It cannot be two different things at the same time. So, we have two types of solutions

$$X_n(x) = \begin{cases} \cos\left(\frac{nx}{2}\right) & n \text{ is odd} \\ \sin\left(\frac{nx}{2}\right) & n \text{ is even.} \end{cases}$$

Here we have

$$\sqrt{|\lambda_n|} = \frac{n}{2}, \quad \lambda_n = -\frac{n^2}{4}.$$

Exercise 1. *Compute that:*

$$\int_{-\pi}^{\pi} \cos(\sqrt{|\lambda_n|}x)^2 dx = \pi = \int_{-\pi}^{\pi} \sin(\sqrt{|\lambda_n|}x)^2 dx.$$

Now we solve for the partner functions,

$$\frac{T_n'}{T_n} = \lambda_n \implies T_n(t) = e^{\lambda_n t}.$$

We ignore the constant factors because they come in at the end. Then, we write

$$u(x,t) = \sum_{n \in \mathbb{N}} T_n(t) X_n(x) \frac{\langle v, X_n \rangle}{||X_n||^2},$$

where

$$v(x) := u(x,0).$$

You can compute these Fourier coefficients if you want to do it, but it's not actually necessary to do it on the exam. Just a friendly little tip for saving time.

1.1.2. *Dealing with time independent inhomogeneities.* Let's consider the problem

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

$$u(-\pi,t) = u(\pi,t) = 0$$

$$u_t(x,0) = 0, \quad x \in [-\pi,\pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 5 \quad x \in [-\pi,\pi], \quad t > 0.$$

OH NO! It's not a homogeneous PDE! What do we do?!?!? Don't panic. Observe that the inhomogeneity is *independent of* t.

> **Idea:** Deal with time independent inhomogeneity in the PDE by finding a steady state solution.

We look for a function $f(x)$ which depends only on x which satisfies the boundary conditions and also satisfies the inhomogeneous PDE. Since f only depends on x, the PDE for f is

$$-f''(x) = 5 \iff f''(x) = -5.$$

This means that

$$f'(x) = -5x + b \implies f(x) = -\frac{5x^2}{2} + bx + c.$$

Now, we want f to satisfy the boundary conditions. So, we want

$$-\frac{5\pi^2}{2} - b\pi + c = 0 = -\frac{5\pi^2}{2} + b\pi + c.$$

If we subtract these equations, then we see that we need to have $b = 0$. If we add these equations then we see that we need

$$-5\pi^2 + 2c = 0 \implies c = \frac{5\pi^2}{2}.$$

Thus, we have found a solution to

$$-f''(x) = 5, \quad f(\pm\pi) = 0,$$

which is

$$f(x) = -\frac{5x^2}{2} + \frac{5\pi^2}{2}.$$

If we then look for a solution to

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases} =: v(x)$$

$$u(-\pi, t) = u(\pi, t) = 0$$
$$u_t(x, 0) = 0, \quad x \in [-\pi, \pi]$$
$$u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad x \in [-\pi, \pi], \quad t > 0,$$

and we add it to f, we will get

$$u(x, 0) + f(x) = v(x) + f(x) \neq v(x).$$

The initial condition gets messed up because of f. So, we need to compensate for this. For that reason, we look for a solution to

$$u(x, 0) = -f(x) + v(x)$$
$$u(-\pi, t) = u(\pi, t) = 0$$
$$u_t(x, 0) = 0, \quad x \in [-\pi, \pi]$$
$$u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad x \in [-\pi, \pi], \quad t > 0.$$

Then, our full solution will be

$$U(x, t) = u(x, t) + f(x).$$

It will now satisfy *everything.* Here it is important to note that when we add u and f, the boundary condition still holds. So, please think about this, because in certain variations on the theme, it could possibly not be true.

Anyhow, we are now just dealing with this nice IVP for the homogeneous wave equation. We can recycle our work from the previous problem. We had the heat equation there, but watch what happens when we separate variables:

$$T''(t)X(x) - X''(x)T(t) = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \lambda,$$

is a constant. So, for the X part, we have the problem:

$$X'' = \lambda X, \quad X(\pm\pi) = 0.$$

Since we have just solved this, we can *skip to the good bit*[1]

$$X_n(x) = \begin{cases} \cos\left(\frac{nx}{2}\right) & n \text{ is odd} \\ \sin\left(\frac{nx}{2}\right) & n \text{ is even.} \end{cases}$$

Here we have

$$\sqrt{|\lambda_n|} = \frac{n}{2}, \quad \lambda_n = -\frac{n^2}{4}.$$

The partner functions,

$$T_n(t) = \alpha_n \cos(\sqrt{|\lambda_n|}x) + \beta_n \sin(\sqrt{|\lambda_n|}x).$$

We shall determine the coefficients using the IC. First, we write

$$u(x, t) = \sum_{n \geq 1} T_n(t) X_n(x).$$

Next, we use the easier of the two ICs, which is

$$u_t(x, 0) = 0.$$

So, we also compute

$$u_t(x, t) = \sum_{n \geq 1} T_n'(t) X_n(x).$$

[1]Recommended listening is the song by Rizzle Kicks.

When we plug in 0, we need to have

$$u_t(x,0) = \sum_{n\geq 1} T_n'(0)X_n(x) = 0.$$

So, to get this, we need

$$T_n'(0) = 0 \forall n.$$

By definition of the T_n,

$$T_n'(0) = \beta_n\sqrt{|\lambda_n|}.$$

So, to make this zero, since $\sqrt{|\lambda_n|} \neq 0$, we need

$$\beta_n = 0 \forall n.$$

Hence, our solution looks like

$$u(x,t) = \sum_{n\geq 1} \alpha_n \cos(\sqrt{|\lambda_n|}t)X_n(x).$$

The other IC says

$$u(x,0) = -f(x) + v(x).$$

Since $\cos(0) = 1$, we see that we need

$$-f(x) + v(x) = \sum_{n\geq 1} \alpha_n X_n(x).$$

This means that we need

$$\alpha_n = \frac{\langle -f + v, X_n\rangle}{||X_n||^2} = \frac{\int_{-\pi}^{\pi} \left(-f(x) + v(x)\right) X_n(x)dx}{\int_{-\pi}^{\pi} |X_n(x)|^2 dx}.$$

It suffices to just leave α_n like this. As we observed before, our full solution is now

$$U(x,t) = u(x,t) + f(x) = -\frac{5x^2}{2} + \frac{5\pi^2}{2} + \sum_{n\geq 1} \alpha_n \cos(\sqrt{|\lambda_n|}x)X_n(x),$$

with X_n defined as above.

1.1.3. *Dealing with non-self-adjoint BCs.* Let's say we have the problem

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 20. \end{aligned}$$

These are *not self adjoint BCs.* Yikes! However, we can use a similar "steady state" trick to deal with this. If the BC $u(0,t) = 20$ were instead $u(0,t) = 0$, then the BCs would be self adjoint BCs. So we want to make it so. Since the PDE is homogeneous, the

Idea: Deal with non-self adjoint BCs which are independent of time by finding a steady state solution.

So, we want a function $f(x)$ which satisfies the equation

$$-f''(x) = 0,$$

and which gives us the bad BC

$$f(0) = 20.$$

We have a nice homogeneous BC on the other side, so we don't want to mess that up, so we want

$$f'(4) = 0.$$

Then, the function

$$f(x) = ax + b.$$

We use the BCs to compute

$$f(0) = 20 \implies b = 20.$$

$$f'(4) = 0 \implies a = 0.$$

Similar to before, if we add it to the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

it's going to screw up the IC. So, instead we look for the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x) - f(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

We can now deal with this in the standard way. We use SV to write $u = XT$ (just a means to an end).[2] Next, we get the equation

$$\frac{T'}{T} = \frac{X''}{X} = \lambda.$$

We solve the SLP

$$X'' = \lambda X, \quad X(0) = 0 = X'(4).$$

The reason we know this is an SLP satisfying the hypotheses of the theorem is because we verify that the BC is self-adjoint.

Exercise 2. *Verify that the only solutions for the cases $\lambda \geq 0$ are solutions which are identically zero.*

[2] La fin justifie les moyens by M.C. Solaar is recommended listening.

We only get $\lambda < 0$. Then, the solution is of the form

$$a_n \cos(\sqrt{|\lambda_n|}x) + b_n \sin(\sqrt{|\lambda_n|}x).$$

The BC at 0 tells us that

$$a_n = 0.$$

The BC at 4 tells us that

$$\cos(\sqrt{|\lambda_n|}4) = 0 \implies \sqrt{|\lambda_n|}4 = \frac{2n+1}{2}\pi \implies \sqrt{|\lambda_n|} = \frac{2n+1}{8}\pi.$$

We then also get

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{64}.$$

We shall deal with the coefficients at the very end. So, we set

$$X_n(x) = \sin(\sqrt{|\lambda_n|}x).$$

The partner function

$$\frac{T_n'}{T_n} = \lambda_n \implies T_n(t) = \alpha_n e^{\lambda_n t} = \alpha_n e^{-(2n+1)^2\pi^2 t/64}.$$

We put it all together writing

$$u(x,t) = \sum_{n\geq 1} T_n(t)X_n(x).$$

To make the IC, we need

$$u(x,0) = \sum_{n\geq 1} T_n(0)X_n(x) = v(x) - f(x).$$

Since

$$T_n(0) = \alpha_n,$$

we need

$$\sum_{n\geq 1} \alpha_n X_n(x) = v(x) - f(x).$$

So we want the coefficients to be the Fourier coefficients of $v - f$, thus

$$\alpha_n = \frac{\langle v - f, X_n\rangle}{||X_n||^2} = \frac{\int_0^4 (v(x) - f(x))\,\overline{X_n(x)}dx}{\int_0^4 |X_n(x)|^2 dx}.$$

Our full solution is

$$U(x,t) = u(x,t) + f(x) = 20 + \sum_{n\geq 1} T_n(t)X_n(x).$$

1.2. **Exercises from [1] for the week.**

1.2.1. *To be demonstrated.*

(1) (4.2:5) Solve:
$$u_t = ku_{xx} + e^{-2t}\sin(x),$$
with
$$u(x,0) = u(0,t) = u(\pi,t) = 0.$$

(2) (EO 23) Determine the eigenvalues and eigenfunctions of the SLP:
$$f'' + \lambda f = 0, \quad 0 < x < a,$$
$$f(0) - f'(0) = 0, \quad f(a) + 2f'(a) = 0.$$

(3) (EO 24) Determine the eigenvalues and eigenfunctions of the SLP:
$$-e^{-4x}\frac{d}{dx}\left(e^{4x}\frac{du}{dx}\right) = \lambda u, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u'(1) = 0.$$

(4) (EO 1) A function is 2 periodic with $f(x) = (x+1)^2$ for $|x| < 1$. Expand $f(x)$ in a Fourier series. Search for a 2 periodic solution to the equation
$$2y'' - y' - y = f(x).$$

(5) (4.2.6) Solve:
$$u_t = ku_{xx} + Re^{-ct}, \quad R, c > 0,$$
$$u(x,0) = 0 = u(0,t) = u(l,t).$$
Physically this is heat flow in a rod which has a chemical reaction in it such that the reaction produced inside the rod dies out over time.

(6) (4.3.5) Find the general solution of
$$u_{tt} = c^2 u_{xx} - a^2 u,$$
$$u(0,t) = u(l,t) = 0,$$
with arbitrary initial conditions. Physically, this is a model for a string vibrating in an elastic medium where the term $-a^2u$ represents the force of reaction of the medium on the string.

1.2.2. *To solve oneself.*

(1) (EO 25) Solve the problem:
$$u_{xx} + u_{yy} = y, \quad 0 < x < 2, \quad 0 < y < 1$$
$$u(x,0) = 0, \quad u(x,1) = 0$$
$$u(0,y) = y - y^3, \quad u(2,y) = 0.$$

(2) (EO 27) Solve the problem
$$u_{xx} + u_{yy} + 20u = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(0,y) = u(1,y) = 0$$
$$u(x,0) = 0, \quad u(x,1) = x^2 - x.$$

(3) (4.4:1) Solve the equation
$$u_{xx} + u_{yy} = 0$$
inside the square $0 \le x, y \le l$, subject to the boundary conditions:
$$u(x,0) = u(0,y = u(l,y) = 0, \quad u(x,l) = x(l-x).$$

(4) (EO 3) Expand the function $\cos(x)$ in a sine series on the interval $(0, \pi/2)$. Use the result to compute

$$\sum_{n\geq 1} \frac{n^2}{(4n^2-1)^2}.$$

(5) (4.2.2) Solve:

$$u_t = ku_{xx}, \quad u(x,0) = f(x),$$
$$u(0,t) = C \neq 0, \quad u_x(l,t) = 0.$$

(6) (4.3.1) Show that the function

$$b_n(t) := \frac{1}{n\pi c}\int_0^t \sin\frac{n\pi c(t-s)}{l}\beta_n(s)ds$$

solves the differential equation:

$$b_n''(t) + \frac{n^2\pi^2c^2}{l^2}b_n(t) = \beta_n(t),$$

as well as the initial conditions $b_n(0) = b_n'(0) = 0$.

(7) (4.4.7) Solve the Dirichlet problem:

$$u_{xx} + u_{yy} = 0 \text{ in } S = \{(r,\theta) : 0 < r_0 \leq r \leq 1, \quad 0 \leq \theta \leq \beta\},$$
$$u(r_0,\theta) = u(1,\theta) = 0, \quad u(r,0) = g(r), \quad u(r,\beta) = h(r).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).