

FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications*, by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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1.1. **Motivation for the Dirichlet problem.** Let us briefly enter two space dimensions. We will do a rather interesting example, solving

$$\square u = 0 \text{ inside a rectangle, } u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = 0,$$

$$u(x, y, t) = g(x, y) \text{ for } (x, y) \text{ on the boundary of the rectangle.}$$

Here we have horrible inhomogeneous boundary condition. So, to solve the problem, we break it into two smaller problems which we tackle *one at a time*: divide and conquer.

Idea: deal with time independent boundary condition by finding a steady state solution.

So, we begin by looking for

$$\Phi(x, y)$$

to satisfy

$$\square \Phi = 0 \text{ inside the rectangle,}$$

$$\Phi = g \text{ on the boundary of the rectangle.}$$

Since the physical problem doesn't care where in space the rectangle is sitting, let us put it so that its vertices are at $(0, 0)$, $(0, B)$, $(A, 0)$, (A, B) .

Idea: deal with each inhomogeneous boundary component one at a time.

It is the same principle: divide and conquer. So, first, let us make nice zero boundary conditions on the sides, and just deal with the complicated boundary conditions on the top and bottom. Therefore we look for a function $\phi(x, y)$ which satisfies

$$\square \phi = 0,$$

$$\phi(0, y) = \phi(A, y) = 0,$$

$$\phi(x, 0) = g(x, 0), \quad \phi(x, B) = g(x, B).$$

Idea: since the PDE is homogeneous and half of the BCs are good and homogeneous, use separation of variables.

We therefore write the PDE:

$$-X''Y - Y''X = 0 \implies -\frac{Y''}{Y} = \frac{X''}{X} = \lambda.$$

The BCs for X are $X(0) = X(A) = 0$. We have solved this problem. The solutions are, up to constant factors

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right), \quad \lambda_n = -\frac{n^2\pi^2}{A^2}.$$

The equation for the partner function is then:

$$-\frac{Y_n''}{Y_n} = \lambda_n \implies Y_n'' = \frac{n^2\pi^2}{A^2}Y_n.$$

A basis of solutions is given by real exponentials, or equivalently hyperbolic sines and cosines. Since our region contains 0, we have been given a hint that using the hyperbolic sines and cosines may be more simple. So, we follow that hint, with

$$Y_n(y) = a_n \cosh\left(\frac{n\pi y}{A}\right) + b_n \sinh\left(\frac{n\pi y}{A}\right).$$

Next we use superposition to create a super solution, which is legit because the PDE is homogeneous:

$$\phi(x, y) = \sum_{n \geq 1} X_n(x)Y_n(y).$$

To obtain the boundary conditions, we need

$$\phi(x, 0) = g(x, 0) = \sum_{n \geq 1} a_n X_n(x).$$

Hence, the coefficients

$$a_n = \frac{\langle g(x, 0), X_n \rangle}{\|X_n\|^2} = \frac{\int_0^A g(x, 0) \overline{X_n(x)} dx}{\int_0^A |X_n(x)|^2 dx}.$$

For the other BC, we need

$$\phi(x, B) = g(x, B) = \sum_{n \geq 1} X_n(x) \left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right) \right).$$

Therefore we need

$$\begin{aligned} \left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right) \right) &= \frac{\langle g(x, B), X_n \rangle}{\|X_n\|^2} \\ &= \frac{\int_0^A g(x, B) X_n(x) dx}{\int_0^A |X_n(x)|^2 dx}. \end{aligned}$$

Solving for b_n we get

$$b_n = \frac{1}{\sinh\left(\frac{n\pi B}{A}\right)} \left(\frac{\langle g(x, B), X_n \rangle}{\|X_n\|^2} - a_n \cosh\left(\frac{n\pi B}{A}\right) \right).$$

Next, we proceed similarly by searching for a function to fix up the BCs on the left and the right. Having dealt with the inhomogeneous BCs at the top and bottom, we set the BC here equal to zero. In that way, when we sum, we shall not mess up the function ϕ . So, we look for a solution to:

$$\square\psi(x, y) = 0, \quad \psi(x, 0) = \psi(x, B) = 0, \quad \psi(0, y) = g(0, y), \quad \psi(A, y) = g(A, y).$$

By symmetry, the solution will be given by

$$\sum_{n \geq 1} \widetilde{X}_n(y) \widetilde{Y}_n(x),$$

with

$$\widetilde{X}_n(y) = \sin\left(\frac{n\pi y}{B}\right),$$

and

$$\widetilde{Y}_n(x) = \widetilde{a}_n \cosh\left(\frac{n\pi x}{B}\right) + \widetilde{b}_n \sinh\left(\frac{n\pi x}{B}\right).$$

The coefficients come from the boundary conditions:

$$\widetilde{a}_n = \frac{\langle g(0, y), \widetilde{X}_n \rangle}{\|\widetilde{X}_n\|^2} = \frac{\int_0^B g(0, y) \widetilde{X}_n(y) dy}{\int_0^B |\widetilde{X}_n(y)|^2 dy}.$$

The other one

$$\widetilde{b}_n = \frac{1}{\sinh\left(\frac{n\pi A}{B}\right)} \left(\frac{\langle g(A, y), \widetilde{X}_n \rangle}{\|\widetilde{X}_n\|^2} - \widetilde{a}_n \cosh\left(\frac{n\pi A}{B}\right) \right).$$

So, we have found

$$\psi(x, y) = \sum_{n \geq 1} \widetilde{X}_n(y) \widetilde{Y}_n(x).$$

The full solution to this part of the problem is

$$\Phi(x, y) = \phi(x, y) + \psi(x, y).$$

Exercise 1. Verify that this function satisfies both the PDE $\square\Phi = 0$ as well as all of the boundary conditions.

To complete the problem, we have only to solve the homogeneous wave equation with the lovely Dirichlet boundary condition and the initial condition with Φ subtracted. So, we are solving:

$$\square u = 0, \quad u_t(x, y, 0) = 0, \quad u(x, y, 0) = f(x, y) - \Phi(x, y), \quad u = 0 \text{ on the boundary.}$$

Idea: since we have homogeneous PDE and BC, use separation of variables and superposition.

We use separation of variables for t , x , and y . Write

$$u = TXY.$$

The PDE is

$$T''XY - X''TY - Y''TX = 0 \iff \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda.$$

Since we have nice homogeneous (Dirichlet) boundary conditions, we begin with the functions that depend on the position in the rectangle, that is X and Y .

Their equation is:

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda \implies \frac{X''}{X} = \lambda - \frac{Y''}{Y}.$$

OBS! The left and right sides depend on *different independent variables*. Hence, by the same reasoning that gave us λ , we get that

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu.$$

Let us solve for X first.¹ So, we are looking to solve:

$$X'' = \mu X, \quad X(0) = X(A) = 0.$$

We have solved this before. The solutions are up to constant factors:

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right) \quad \mu_n = -\frac{n^2\pi^2}{A^2}.$$

This gives the equation for Y ,

$$\frac{Y''}{Y} = \lambda - \mu_n, \quad Y(0) = Y(B) = 0.$$

Let us briefly call

$$\nu = \lambda - \mu_n.$$

Then, this is just the same equation but with different names for things:

$$Y'' = \nu Y, \quad Y(0) = Y(B) = 0.$$

Up to constant factors, the solutions are

$$Y_m(y) = \sin\left(\frac{m\pi y}{B}\right) \quad \nu_m = -\frac{m^2\pi^2}{B^2}.$$

Since

$$\nu_m = \lambda - \mu_n \implies \lambda = \lambda_{n,m} = \nu_m + \mu_n = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}.$$

Recalling the equation for the partner function, T , we have

$$T_{n,m}(t) = a_{n,m} \cos(\sqrt{|\lambda_{n,m}|}t) + b_{n,m} \sin(\sqrt{|\lambda_{n,m}|}t).$$

Hence we write

$$u(x, y, t) = \sum_{n,m \geq 1} T_{n,m}(t) X_n(x) Y_m(y).$$

The initial condition

$$u_t(x, y, 0) = 0 \implies b_{n,m} = 0 \forall n, m.$$

The other condition is that

$$u(x, y, 0) = f(x, y) - \Phi(x, y) = \sum_{n,m \geq 1} a_{n,m} X_n(x) Y_m(y).$$

Hence we require

$$a_{n,m} = \frac{\langle f - \Phi, X_n Y_m \rangle}{\|X_n Y_m\|^2} = \frac{\int_{[0,A] \times [0,B]} (f(x, y) - \Phi(x, y)) X_n(x) Y_m(y) dx dy}{\int_{[0,A] \times [0,B]} |X_n(x) Y_m(y)|^2 dx dy}.$$

The full solution is then

$$u(x, y, t) - \Phi(x, y).$$

¹In this case, we could solve for either X or Y first, it actually does not matter which you choose.

Remark 1. *The eigenvalues of the two-dimensional SLP we solved above,*

$$\lambda_{n,m} = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}$$

are interesting to compare to the analogous one-dimensional case. In the analogous one dimension case, where we have

$$\mu_n = -\frac{n^2\pi^2}{A^2},$$

you can see that these are all square integer multiples of

$$\mu_1 = -\frac{\pi^2}{A^2}.$$

This is the mathematical reason that vibrating strings sound lovely. On the other hand, as long as the rectangle is not a square, that is $A \neq B$, it is no longer true that the $\lambda_{n,m}$ are all multiples of

$$\lambda_{1,1} = -\frac{\pi^2}{B^2} - \frac{\pi^2}{A^2}.$$

For this reason, vibrating rectangles can sound rather awful. You can listen to something along these lines (okay it's for tori not rectangles, but mathematically basically the same) here: <http://www.toroidalsnark.net/som.html>. Further exploration of the mathematics of music could make for an interesting bachelor's or master's thesis....

1.2. Exercises to solve oneself: hints.

(1) (EO 25) Solve the problem:

$$\begin{aligned} u_{xx} + u_{yy} &= y, & 0 < x < 2, & \quad 0 < y < 1 \\ u(x, 0) &= 0, & u(x, 1) &= 0 \\ u(0, y) &= y - y^3, & u(2, y) &= 0. \end{aligned}$$

Hint: divide and conquer. First, find a function which is independent of x to solve the inhomogeneous PDE. That is you want $f(y)$ to solve:

$$f''(y) = y, \quad f(0) = 0 = f(1).$$

Next solve the problem

$$\begin{aligned} v_{xx} + v_{yy} &= 0, \\ v(x, 0) &= 0 = v(x, 1), \\ v(0, y) &= y - y^3 - f(y), & v(2, y) &= -f(y). \end{aligned}$$

Show that the solution to the original problem is given by

$$u(x, y) = v(x, y) + f(y).$$

(2) (EO 27) Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} + 20u &= 0, & 0 < x < 1, & \quad 0 < y < 1, \\ u(0, y) &= u(1, y) = 0 \\ u(x, 0) &= 0, & u(x, 1) &= x^2 - x. \end{aligned}$$

Hint: divide and conquer. The PDE is homogeneous. So write $u = XY$. Plug it into the PDE. Solve for X function first because it has nice Dirichlet

boundary conditions. Then solve for the partner Y function. Use the condition $u(x, 1) = x^2 - x$ to determine the unknown coefficients.

- (3) (4.4:1) Solve the equation

$$u_{xx} + u_{yy} = 0$$

inside the square $0 \leq x, y \leq l$, subject to the boundary conditions:

$$u(x, 0) = u(0, y) = u(l, y) = 0, \quad u(x, l) = x(l - x).$$

Hint: follow the same procedure as the preceding exercise.

- (4) (EO 3) Expand the function $\cos(x)$ in a sine series on the interval $(0, \pi/2)$. Use the result to compute

$$\sum_{n \geq 1} \frac{n^2}{(4n^2 - 1)^2}.$$

Hint: the coefficients in a sine series will be given by

$$\beta_n = \frac{4}{\pi} \int_0^{\pi/2} \cos(x) \sin(2nx) dx.$$

One way to make this integral easier is to expand stuff into complex exponentials from which you can obtain that

$$\cos(ax) \sin(bx) = \frac{1}{2} (\sin((a + b)x) - \sin((a - b)x)).$$

To compute the big sum at the end, use Parseval's equation.

- (5) (4.2.2) Solve:

$$\begin{aligned} u_t &= k u_{xx}, & u(x, 0) &= f(x), \\ u(0, t) &= C \neq 0, & u_x(l, t) &= 0. \end{aligned}$$

Hint: First deal with that icky inhomogeneous boundary condition $C \neq 0$ by finding a steady state solution as in lecture. This is $\phi(x)$ which has $\phi''(x) = 0$, $\phi(0) = C$, $\phi'(l) = 0$. Then, look for a solution to solve

$$\begin{aligned} u_t &= k u_{xx}, & u(x, 0) &= f(x) - \phi(x), \\ u(0, t) &= 0, & u_x(l, t) &= 0. \end{aligned}$$

For this problem you can now use separation of variables, SLP theory, Hilbert space theory, and finally compute your coefficients using the initial data. It's all coming together!

- (6) (4.3.1) Show that the function

$$b_n(t) := \frac{1}{n\pi c} \int_0^t \sin \frac{n\pi c(t-s)}{l} \beta_n(s) ds$$

solves the differential equation:

$$b_n''(t) + \frac{n^2 \pi^2 c^2}{l^2} b_n(t) = \beta_n(t),$$

as well as the initial conditions $b_n(0) = b_n'(0) = 0$. Hint: Use the fundamental theorem of calculus to compute $b_n'(t)$. Check the initial conditions by just substituting 0 for t . Next, compute the derivative of b_n' to get b_n'' and check the equation.

(7) (4.4.7) Solve the Dirichlet problem:

$$u_{xx} + u_{yy} = 0 \text{ in } S = \{(r, \theta) : 0 < r_0 \leq r \leq 1, \quad 0 \leq \theta \leq \beta\},$$

$$u(r_0, \theta) = u(1, \theta) = 0, \quad u(r, 0) = g(r), \quad u(r, \beta) = h(r).$$

Hint: Turn the equation into polar coordinates. An annulus looks like a rectangle in polar coordinates. Next separate variables and write $u = R(r)\Theta(\theta)$. Solve for the R function first because it has the beautiful boundary conditions. This is going to become an Euler equation like we solved for in lecture. So, check your lecture notes to see how we did that (Day 9). Once you have your R functions, they will be like $R_n(r)$, with corresponding λ_n , use this to find the partner Θ_n functions. Finally use g and h to determine the unknown coefficients. This part is a bit like finding the coefficients when solving the Dirichlet problem in a rectangle.

REFERENCES

- [1] Gerald B. Folland, *Fourier Analysis and Its Applications*, Pure and Applied Undergraduate Texts Volume 4, (1992).