FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.02.18

Theorem 1 (Properties of the Fourier transform). *Assume that everything below is well defined. Then, the Fourier transform,*

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

satisfies

(1) $\mathcal{F}(f(x-a))(\xi) = e^{-ia}\hat{f}(\xi)$.
(2) $\mathcal{F}(f')(\xi) = i\xi\hat{f}(\xi)$
(3) $\mathcal{F}(xf(x))(\xi) = i\mathcal{F}(f)'(\xi)$
(4) $\mathcal{F}(f*g)(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

Proof: We just compute (we are being a bit naughty, not bothering with issues of convergence, but all such issues are indeed rigorously verifiable, so not to worry). First

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(x-a)e^{-ix\xi}dx.$$

Change variables. Let $t = x - a$, then $dt = dx$, and $x = t + a$ so

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(t)e^{-i(t+a)\xi}dt = e^{-ia\xi}\hat{f}(\xi).$$

The next one will come from integrating by parts:

$$\int_{\mathbb{R}} f'(x)e^{-ix\xi}dx = f(x)e^{-ix\xi}\big|_{-\infty}^{\infty} - \int_{\mathbb{R}} -i\xi f(x)e^{-ix\xi}dx = i\xi\hat{f}(\xi).$$

The boundary terms vanish because of reasons (again it is $\mathcal{L}^1$ and $\mathcal{L}^2$ theory stuff). Similarly we compute

$$\int_{\mathbb{R}} xf(x)e^{-ix\xi}dx = -\frac{1}{i}\int_{\mathbb{R}} f(x)\frac{d}{d\xi}e^{-ix\xi}dx = i\frac{d}{d\xi}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx = i\mathcal{F}(f)'(\xi).$$

Finally,

$$\mathcal{F}(f*g)(\xi) = \int_{\mathbb{R}} f*g(x)e^{-ix\xi}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-ix\xi}dydx.$$

We do a little sneaky trick

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(y) e^{-ix\xi} e^{-iy\xi} e^{iy\xi} dy dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) e^{-i(x-y)\xi} g(y) e^{-iy\xi} dy dx.$$

Let $z = x - y$. Then $dz = -dy$ so

$$= \int_{\mathbb{R}} \int_{\infty}^{-\infty} f(z) e^{-iz\xi} (-dz) g(y) e^{-iy\xi} dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) e^{-iz\xi} dz g(y) e^{-iy\xi} dy$$

$$= \hat{f}(\xi) \hat{g}(\xi).$$

It shall be quite useful to know how to "undo" the Fourier transform.

Theorem 2 (Extension of Fourier transform to $\mathcal{L}^2$)**.** *There is a well defined unique extension of the Fourier transform to $\mathcal{L}^2(\mathbb{R})$. The Fourier transform of an element of $\mathcal{L}^2(\mathbb{R})$ is again an element of $\mathcal{L}^2(\mathbb{R})$. Moreover, for any $f \in \mathcal{L}^2(\mathbb{R})$ we have the FIT (Fourier Inversion Theorem):*

eq:fit (1.1)
$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi.$$

The theory item FIT is a *Julklapp.* All you need to know is the equation (eq:fit 1.1). The next theorem is also a theory item, with a short proof. The key is to start on the left side and use the FIT.

Theorem 3 (Plancharel)**.** *For any $f \in \mathcal{L}^2(\mathbb{R})$, $\hat{f} \in \mathcal{L}^2(\mathbb{R})$. Moreover,*

$$\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle,$$

and thus

$$||\hat{f}||^2_{\mathcal{L}^2} = 2\pi ||f||^2,$$

for all f and g in $\mathcal{L}^2(\mathbb{R})$.

Proof: Start with the right side and use the FIT on f, to write

$$2\pi \langle f, g \rangle = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi} e^{ix\xi} \hat{f}(\xi) \overline{g(x)} d\xi dx = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \overline{g(x)} d\xi dx.$$

Move the complex conjugate to engulf the $e^{ix\xi}$,

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \overline{g(x) e^{-ix\xi}} d\xi dx.$$

Swap the order of integration and integrate x first:

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \overline{g(x) e^{-ix\xi}} dx d\xi = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \langle \hat{f}, \hat{g} \rangle.$$

We may from time to time use the following cute fact as well.

Lemma 4 (Riemann & Lebesgue)**.** *Assume $f \in \mathcal{L}^1(\mathbb{R})$. Then,*

$$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0.$$

We shall indeed need to actually prove the next one, because it's going to be quite important for the initial value problem for the heat equation.

1.1. **The big bad convolution approximation theorem.**

This theory item is Theorem 7.3, regarding approximation of a function by convoluting it with a so-called "approximate identity." This theorem and its proof are both rather long. The proof relies very heavily on knowing the definition of limits and how to work with those definitions, so if you're not comfortable with ϵ and δ style arguments, it would be advisable to brush up on these.

Theorem 5. *Let* $g \in L^1(\mathbb{R})$ *such that*

$$\int_{\mathbb{R}} g(x)dx = 1.$$

Define

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f *is piecewise continuous on* $\mathbb{R}$ *and its left and right sided limits exist for all points of* $\mathbb{R}$*. Assume that either* f *is bounded on* $\mathbb{R}$ *or that* g *vanishes outside of a bounded interval. Let, for* $\varepsilon > 0$*,*

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof. **Idea 1:** Do manipulations to get a "left side" statement and a "right side" statement.

We would like to show that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy = \alpha f(x+) + \beta f(x-)$$

which is equivalent to showing that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \alpha f(x+) - \beta f(x-) = 0.$$

We now insert the definitions of α and β, so we want to show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

We can prove this if we show that

$$\text{left side } \lim_{\varepsilon \to 0} \int_{-\infty} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy = 0$$

and also

$$\text{right side } \lim_{\varepsilon \to 0} \int_{0}^{\infty} f(x-y) g_\varepsilon(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

In the textbook, Folland proves that the second of these (right side) holds. So, for the sake of diversity, we prove that the first of these holds (left side). The argument is the same for both, so proving one of them is sufficient.

Hence, we would like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy$$

as small as we like. To make this precise, let us assume that "as small as we like" is quantified by a very small $\delta > 0$. Then we show that for sufficiently small ε we obtain

$$\left| \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy \right| < \delta.$$

Idea 2: Smash the two integrals together:

$$\int_{-\infty}^{0} \left(f(x-y)g_\varepsilon(y) - f(x+)g(y) \right) dy.$$

Well, this is a bit inconvenient, because in the first part we have g_ε, but in the second part it's just g.

Idea 3: Sneak g_ε into the second term. We make a small observation,

$$\int_{-\infty}^{0} g(y)dy = \int_{-\infty}^{0} g(z/\varepsilon)\frac{dz}{\varepsilon} = \int_{-\infty}^{0} g_\varepsilon(z)dz$$

Above, we have made the substitution $z = \varepsilon y$, so $y = z/\varepsilon$, and $dz/\varepsilon = dy$. The limits of integration don't change. By this calculation,

$$\int_{-\infty}^{0} f(x+)g(y)dy = \int_{-\infty}^{0} f(x+)g_\varepsilon(y)dy.$$

(Above the integration variable was called z, but what's in a name? The name of the integration variable doesn't matter!). Moreover, note that $f(x+)$ is a constant, so it's just sitting there doing nothing. Hence, we have computed that

$$\int_{-\infty}^{0} \left(f(x-y)g_\varepsilon(y) - f(x+)g(y) \right) dy = \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy.$$

Remember that $y \le 0$ where we're integrating. Therefore, $x - y \ge x$.

Idea 4: Use the definition of right hand limit:

$$\lim_{y \uparrow 0} f(x-y) = f(x+) \implies \lim_{y \uparrow 0} f(x-y) - f(x+) = 0.$$

By the definition of limit there exists $y_0 < 0$ such that for all $y \in (y_0, 0)$

$$|f(x-y) - f(x+)| < \widetilde{\delta}.$$

We are using $\widetilde{\delta}$ for now, to indicate that $\widetilde{\delta}$ is going to be something in terms of δ, engineered in such a way that at the end of our argument we get that for ϵ sufficiently small,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right| < \delta.$$

To figure out this $\widetilde{\delta}$, we use our estimate on the part of the integral from y_0 to 0,

$$\left| \int_{y_0}^{0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \le \int_{y_0}^{0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy$$

$$\le \widetilde{\delta} \int_{y_0}^{0} |g_\varepsilon(y)|dy \le \widetilde{\delta} \int_{\mathbb{R}} |g_\varepsilon(y)|dy = \widetilde{\delta}||g||.$$

Above, we have used the same substitution trick to see that

$$\int_{\mathbb{R}} |g_\varepsilon(y)|dy = \int_{\mathbb{R}} |g(z)|dz = ||g||,$$

where $||g||$ is the $L^1(\mathbb{R})$ norm of g. By assumption, $g \in L^1(\mathbb{R})$, so this L^1 norm is finite. Moreover, because we know that

$$\int_{\mathbb{R}} g(y)dy = 1,$$

we know that

$$||g|| = \int_{\mathbb{R}} |g(y)|dy \geq \left| \int_{\mathbb{R}} g(y)dy \right| = 1.$$

So, let

$$\widetilde{\delta} = \frac{\delta}{2||g||}.$$

Note that we're not dividing by zero, by the above observation that $||g|| \geq 1$. So, this is a perfectly decent number. Then, we have the estimate (repeating the above estimate)

$$\left| \int_{y_0}^{0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \leq \int_{y_0}^{0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy$$

$$\leq \widetilde{\delta} \int_{y_0}^{0} |g_\varepsilon(y)|dy \leq \widetilde{\delta} \int_{\mathbb{R}} |g_\varepsilon(y)|dy = \widetilde{\delta}||g|| = \frac{\delta}{2}.$$

Idea 5: To deal with the other part of the integral, from $-\infty$ to y_0, consider the two cases given in the statement of the theorem separately. It is important to remember that

$$y_0 < 0.$$

So, we wish to estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right|.$$

First, let us assume that f is bounded, which means that there exists $M > 0$ such that $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$. Hence

$$|f(x-y) - f(x+)| \leq |f(x-y)| + |f(x+)| \leq 2M.$$

So, we have the estimate

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \leq \int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy \leq 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)|dy.$$

We shall do a substitution now, letting $z = y/\varepsilon$. Then, as we have computed before,

$$\int_{-\infty}^{y_0} |g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon} |g(z)|dz.$$

Here the limits of integration do change, because $y_0 < 0$. Specifically $y_0 \neq 0$, which is why the top limit changes. We're integrating between $-\infty$ and y_0/ε. We know that $y_0 < 0$. So, when we divide it by a really small, but still positive number, like ε, then $y_0/\varepsilon \to -\infty$ as $\varepsilon \to 0$. Moreover, we know that

$$\int_{-\infty}^{0} |g(y)|dy < \infty.$$

What this really means is that

$$\lim_{R\to-\infty}\int_R^0 |g(y)|dy = \int_{-\infty}^0 |g(y)|dy < \infty.$$

Hence,

$$\lim_{R\to-\infty}\int_{-\infty}^0 |g(y)|dy - \int_R^0 |g(y)|dy = 0.$$

Of course, we know what happens when we subtract the integral, which shows that

$$\lim_{R\to-\infty}\int_{-\infty}^R |g(y)|dy = 0.$$

Since

$$\lim_{\varepsilon\to 0} y_0/\varepsilon = -\infty,$$

this shows that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{y_0/\varepsilon} |g(y)|dy = 0.$$

Hence, by definition of limit there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{-\infty}^{y_0/\varepsilon} |g(y)|dy < \frac{\delta}{4(M+1)}.$$

Then, combining this with our estimates, above, which we repeat here,

$$\left|\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| \le \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy \le 2M\int_{-\infty}^{y_0} |g_\varepsilon(y)|dy$$

$$< 2M\frac{\delta}{4(M+1)} < \frac{\delta}{2}.$$

Therefore, we have the estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left|\int_{-\infty}^0 g_\varepsilon(y)\left(f(x-y) - f(x+)\right)dy\right|$$

$$\le \int_{-\infty}^0 |g_\varepsilon(y)||f(x-y)-f(x+)|dy \le \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^0 |f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Finally, we consider the other case in the theorem, which is that g vanishes outside a bounded interval. We retain the first part of our estimate, that is

$$\int_{y_0}^0 |f(x-y) - f(x+)||g_\varepsilon(y)|dy < \frac{\delta}{2}.$$

Next, we again observe that

$$\lim_{\varepsilon\downarrow 0}\frac{y_0}{\varepsilon} = -\infty.$$

By assumption, we know that there exists some $R > 0$ such that

$$g(x) = 0 \forall x \in \mathbb{R} \text{ with } |x| > R.$$

Hence, we may choose ε sufficient small so that

$$\frac{y_0}{\varepsilon} < -R.$$

Specifically, let

$$\varepsilon_0 = \frac{1}{-Ry_0} > 0.$$

Then for all $\varepsilon \in (0, \varepsilon_0)$ we compute that

$$\frac{y_0}{\varepsilon} < -R.$$

Hence for all $y \in (-\infty, y_0/\varepsilon)$ we have $g(y) = 0$. Thus, we compute as before using the substitution $z = y/\varepsilon$,

$$\int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon} |f(x-\varepsilon z) - f(x+)||g(z)|dz = 0,$$

because $g(z) = 0 \forall z \in (-\infty, y_0/\varepsilon)$. Thus, we have the total estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right|$$

$$\leq \int_{-\infty}^{0} |g_\varepsilon(y)||f(x-y)-f(x+)|dy \leq \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$< 0 + \frac{\delta}{2} \leq \delta.$$

□

As a consequence we can obtain a more robust version of the Fourier inversion theorem.

Theorem 6 (Fourier inversion theorem)**.** *Assume that $f \in \mathcal{L}^1(\mathbb{R})$ and is piecewise continuous on $\mathbb{R}$. Assume that at its points of discontinuity*

$$f(x) = \frac{1}{2} \left(f(x_-) + f(x_+) \right).$$

Then

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi.$$

Moreover, if $\hat{f} \in \mathcal{L}^1(\mathbb{R})$, then f is continuous and

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi.$$

Finally, if $f \in \mathcal{L}^2(\mathbb{R})$, then the equality

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi$$

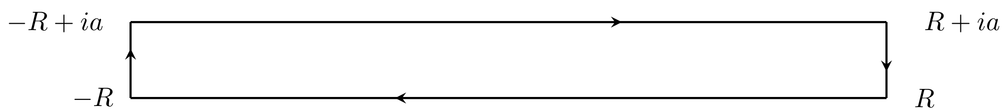
holds for almost every $x \in \mathbb{R}$.

Proof: The first part of the theorem is an application of the big bad convolution theorem. Let's just write out the integral on the right

$$\int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi.$$

Using the definition of the Fourier transform, this is

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} e^{-iy\xi} e^{-\varepsilon^2 \xi^2/2} f(y) dy d\xi = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi(x-y)} e^{-\varepsilon^2 \xi^2/2} f(y) dy d\xi.$$



box FIGURE 1. The contour over which we integrate. Call the contour Γ.

If we (to the chagrin of the theoretical mathematicians, but again, trust me, it's rigorously justifiable!) change the order of integration and do the ξ integral first, we are computing the Fourier transform of $e^{-\varepsilon^2\xi^2/2}$ evaluated at the point $y-x$. The idea is to complete the square in the exponent:

$$-\varepsilon^2\xi^2/2 + i\xi(x-y) = -\left(\frac{\varepsilon\xi}{\sqrt{2}} - \frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2 + \left(\frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2.$$

Therefore

$$\int e^{-\varepsilon^2\xi^2/2+i\xi(x-y)}d\xi = \int e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}} - \frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2 + \left(\frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2} d\xi$$
$$= e^{-\frac{(x-y)^2}{2\varepsilon^2}} \int e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}} - \frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2} d\xi.$$

Using a teensy bit of complex analysis, we can throw away the imaginary part in the exponent. To see this, draw a box, as in Figure 1 and integrate over it. The integrals on the sides tend to zero, and the function is holomorphic inside. Thus the contour integral is zero, so we get that the integral on the top and bottom sides must cancel. Hence:

Rbox
$$\lim_{R\to\infty}\int_{-R}^{R} e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}} - \frac{i(x-y)}{\varepsilon\sqrt{2}}\right)^2} d\xi = \lim_{R\to\infty}\int_{-R}^{R} e^{-\left(\frac{\varepsilon\xi}{\sqrt{2}}\right)^2} d\xi. \tag{1.2}$$

For the sake of simplicity, assume $y > x$. Let

$$a = \frac{(y-x)}{\varepsilon^2}.$$

We integrate the function

$$f(z) = e^{-\frac{\varepsilon^2}{2}z^2}.$$

At the bottom of the box, $z = \xi \in \mathbb{R}$, and

$$f(z) = f(\xi) = e^{-\frac{\varepsilon^2\xi^2}{2}}.$$

At the top of the box $z = \xi + ia$, where

$$a = \frac{(y-x)}{\varepsilon^2} \implies f(z) = f(\xi+ia) = \exp\left(-\frac{\varepsilon^2}{2}\left(\xi + i\frac{(y-x)}{\varepsilon^2}\right)^2\right)$$
$$= \exp\left(-\left(\frac{\varepsilon\xi}{\sqrt{2}} - i\frac{(x-y)}{\sqrt{2}\varepsilon}\right)^2\right).$$

Since the function $f(z)$ is super nice and holomorphic,

$$\int_\Gamma f(z)dz = 0.$$

Moreover, the left and right integrals have $|\Re(z)| = R$. So, there,

$$f(\pm R + i\Im z) = \exp\left(-\frac{\varepsilon^2}{2}(\pm R + i\Im z)^2\right) \le \exp\left(-\frac{\varepsilon^2 R^2}{2}\right),$$

since $\Im z \geq 0$. Hence, the integral on the two sides are bounded above by

$$|a| \exp\left(-\frac{\varepsilon^2 R^2}{2}\right) \to 0 \text{ as } R \to \infty.$$

So, as we let $R \to \infty$, the integral over the contour remains zero, the integrals over the sides of the box tend to zero, and we indeed see that (1.2) is true.

Thus, we just need to compute

$$\int e^{-\xi^2 \varepsilon^2/2} d\xi.$$

We use a substitution, $t = \xi\varepsilon/\sqrt{2}$ so $dt = \varepsilon d\xi/\sqrt{2}$, and

$$\int e^{-\xi^2\varepsilon^2/2} d\xi = \frac{\sqrt{2}}{\varepsilon} \int e^{-t^2} dt = \frac{\sqrt{2\pi}}{\varepsilon}.$$

Thus

$$\mathcal{F}(e^{-\varepsilon^2\xi^2/2})(y-x) = \int e^{-\varepsilon^2\xi^2/2 + i\xi(x-y)} d\xi = e^{-\frac{(x-y)^2}{2\varepsilon^2}} \frac{\sqrt{2\pi}}{\varepsilon}.$$

So, in summary, we have computed:

$$\int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2\xi^2/2} \hat{f}(\xi) d\xi = \frac{\sqrt{2\pi}}{\varepsilon} \int e^{-\frac{(x-y)^2}{2\varepsilon^2}} f(y) dy = \sqrt{2\pi} \int \frac{e^{-\frac{(x-y)^2}{2\varepsilon^2}}}{\varepsilon} f(y) dy.$$

We computed

$$\int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2\xi^2/2} \hat{f}(\xi) d\xi = \frac{\sqrt{2\pi}}{\varepsilon} \int e^{-\frac{(x-y)^2}{2\varepsilon^2}} f(y) dy = \sqrt{2\pi} \int \frac{e^{-\frac{(x-y)^2}{2\varepsilon^2}}}{\varepsilon} f(y) dy.$$

Ignoring the constant in front, we recognize

$$\int \frac{e^{-\frac{(x-y)^2}{2\varepsilon^2}}}{\varepsilon} f(y) dy = g_\varepsilon \star f(x), \quad g(x) = e^{-\frac{x^2}{2}} \implies g_\varepsilon(x) = \frac{1}{\varepsilon} e^{-\frac{x^2}{2\varepsilon^2}}.$$

Now, we see that, letting $t = x/\sqrt{2}$ so $\sqrt{2}dt = dx$

$$\int_{\mathbb{R}} g(x) dx = \int_{\mathbb{R}} e^{-x^2/2} dx = \int_{\mathbb{R}} e^{-t^2} \sqrt{2} dt = \sqrt{2\pi}.$$

Moreover, g is an even function, so

$$\int_{-\infty}^0 g(x) dx = \frac{\sqrt{2\pi}}{2} = \int_0^\infty g(x) dx.$$

Hence, the big bad convolution theorem says that

$$\lim_{\varepsilon \to 0} g_\varepsilon \star f(x) = \frac{\sqrt{2\pi}}{2} \left(f(x_-) + f(x_+)\right).$$

Therefore when we bring back the constant factor of $\sqrt{2\pi}$, we get

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2\xi^2/2} \hat{f}(\xi) d\xi = \lim_{\varepsilon \to 0} \sqrt{2\pi} g_\varepsilon \star f(x) = \frac{2\pi}{2} \left(f(x_-) + f(x_+)\right).$$

So, dividing by 2π, we get

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2\xi^2/2} \hat{f}(\xi) d\xi = \frac{1}{2} \left(f(x_-) + f(x_+)\right) = f(x).$$

So that's how we get the first part, because at points of continuity, $f(x)$ is the average of its left and right limits, and at points of discontinuity, we assumed it was defined to equal this average. For the second statement, we look at

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon^2 \xi^2/2} \hat{f}(\xi) d\xi.$$

Well, since the integrand is in $\mathcal{L}^1$ we can use the dominated convergence theorem (if we know what the heck that is) to bring the limit into the integral. When we do that we get

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi.$$

We already saw that this limit converges to $f(x)$. However, now to get the continuity we use this integral formula for f and the fact that $\hat{f}$ is in $\mathcal{L}^1$ to use the DCT again (if we know what the heck that is) to say

$$\lim_{\delta \to 0} f(x+\delta) = \lim_{\delta \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(x+\delta)} \hat{f}(\xi) d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \lim_{\delta \to 0} e^{i\xi(x+\delta)} \hat{f}(\xi) d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi$$
$$= f(x).$$

Don't be cross with me for taking the limit into the integral. It is JUSTIFIED by the DCT (dominated convergence theorem). Just rest assured that if you take Integration Theory, you will see that indeed, the above switch-a-roo of limits is rigorously valid.

The last statement for $f \in \mathcal{L}^2(\mathbb{R})$ could be shown using an approximation argument. If it is too theoretical, just ignore this part. Smooth, compactly supported (this means they are zero outside of a compact set) functions are dense in $\mathcal{L}^2$. So, we take a sequence of them ϕ_n with $\phi_n \to f$ in $\mathcal{L}^2$ norm. Smooth, compactly supported functions are Schwarz class, so their Fourier transforms are Schwarz class (hence in $\mathcal{L}^1$. Schwarz class means the function and all its derivatives decay faster than x^{-n} as $|x| \to \infty$ for any $n \in \mathbb{N}$. This is called rapidly decaying). Hence, for each of the ϕ_n, its Fourier transform is in $\mathcal{L}^1$ so we got

$$\phi_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{\phi}_n(x) dx.$$

We have

$$||\phi_n - f|| \to 0, \quad n \to \infty$$

because we assumed $\phi_n \to f$ in $\mathcal{L}^2$ norm. By Plancharel's theorem, we also have

$$||\hat{\phi}_n - \hat{f}|| \to 0, \quad n \to \infty$$

and therefore also

$$\int_{\mathbb{R}} |e^{ix\xi} \hat{\phi}_n(\xi) - e^{ix\xi} \hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\hat{\phi}_n(\xi) - \hat{f}(\xi)|^2 d\xi = ||\hat{\phi}_n - \hat{f}||^2 \to 0, \quad n \to \infty.$$

This shows that

$$||\frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{\phi}_n(\xi) d\xi - \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi|| \to 0, \quad n \to \infty.$$

So, since ϕ_n is given by this integral formula we have

$$\phi_n \to \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi, \quad n \to \infty$$

as elements of $\mathcal{L}^2$, and also

$$\phi_n \to f, \quad n \to \infty$$

as elements of $\mathcal{L}^2$. By the uniqueness of limits in $\mathcal{L}^2$,

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi, \qquad \text{as elements of } \mathcal{L}^2.$$

This means by definition of $\mathcal{L}^2$ that $f(x)$ is equal to that integral on the right for almost every x. That's good enough.

1.2. **Exercises for the week to be demonstrated.** On Monday in the large group we shall have:

(1) (7.2.13.b) Use Plancharel's theorem to compute:

$$\int_{\mathbb{R}} \frac{t^2}{(t^2+a^2)(t^2+b^2)} dt = \frac{\pi}{a+b}.$$

(2) (Eö 12) Let

$$f(t) = \int_0^1 \sqrt{w} e^{w^2} \cos(wt) dw.$$

Compute

$$\int_{\mathbb{R}} |f'(t)|^2 dt.$$

(3) (7.4.1.a,b) Compute the Fourier sine and cosine transforms of e^{-kx}. These are defined, respectively, to be

$$\mathcal{F}_s[f](\xi) = \int_0^\infty f(x) \sin(\xi x) dx, \quad \mathcal{F}_c[f](\xi) = \int_0^\infty f(x) \cos(\xi x) dx.$$

On Wednesday or Friday depending on your group we shall have:

(1) (Eö 6.a, b) Compute the Fourier transforms of:

$$\frac{t}{(t^2+a^2)^2}, \quad \frac{1}{(t^2+a^2)^2}.$$

(2) (Eö 7) A function has Fourier transform

$$\hat{f}(\xi) = \frac{\xi}{1+\xi^4}.$$

Compute

$$\int_{\mathbb{R}} t f(t) dt, \quad f'(0).$$

(3) (7.3.2) Use the Fourier transform to derive the solution of the inhomogeneous heat equation $u_t = k u_{xx} + G(x,t)$ with initial condition $u(x,0) = f(x)$ (assume $f \in \mathcal{L}^2(\mathbb{R})$:

$$u(x,t) = f * K_t(x) + \int_{\mathbb{R}} \int_0^t G(y,s) K_{t-s}(x-y) ds dy.$$

Here

$$K_t(x) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}.$$

1.3. **Exercises for the week to be done oneself.**

(1) (Eö 9) Compute (with help of Fourier transform)
$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)} dx.$$
(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a, b) both directly and by using the known case for the interval $(-a, a)$.
(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \leq 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1 + i\xi)^{-a}$ where Γ is the Gamma function.
(4) (7.2.12) For $a > 0$ let
$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$
Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.
(5) (Eö 6.d,e) Compute the Fourier transform of:
$$e^{-a|t|}\sin(bt), \quad (a, b > 0), \quad \frac{t}{t^2+2t+5}.$$
(6) (Eö 15) Find a solution to the equation
$$u(t) + \int_{-\infty}^{t} e^{\tau - t} u(\tau) d\tau = e^{-2|t|}.$$
(7) (Eö 11) For the function
$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w} e^{iwt} dw,$$
compute
$$\int_{\mathbb{R}} f(t)\cos(t) dt, \quad \int_{\mathbb{R}} |f(t)|^2 dt.$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).