FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Today we will apply the theory to solve:

(1) the homogeneous heat equation
(2) tricky integrals
(3) the inhomogeneous heat equation.

If time allows, we may prove the beautiful Sampling Theorem.

1.1. **Application to IVP for the homogeneous heat equation.** We wish to find u to satisfy

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = v(x) \in \mathcal{L}^2(\mathbb{R}) \end{cases}$$

We hit the PDE with the Fourier transform IN THE x VARIABLE:

$$\hat{u}_t(\xi,t) - \hat{u}_{xx}(\xi,t) = 0.$$

Now, we use the theorem which gave us the properties of the Fourier transform. It says that if we take the Fourier transform of a derivative, $\hat{f'}(\xi) = i\xi\hat{f}(\xi)$. Using this twice,

$$\hat{u}_{xx}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

Now, those of you who are picky about switching limits may not like this, but it is in fact rigorously valid:

$$\partial_t\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t) = 0.$$

Hence

$$\partial_t\hat{u}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

This is a first order homogeneous ODE for u in the t variable. We can solve it!!! We do that and get

$$\hat{u}(\xi,t) = e^{-\xi^2 t}c(\xi).$$

The constant can depend on ξ but not on t. To figure out what the constant should be, we use the IC:

$$\hat{u}(\xi,0) = \hat{v}(\xi) \implies c(\xi) = \hat{v}(\xi).$$

Thus, we have found

$$\hat{u}(\xi,t) = e^{-\xi^2 t}\hat{v}(\xi).$$

Now, we use another property of the Fourier transform which says

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

So, if we can find a function whose Fourier transform is $e^{-\xi^2 t}$, then we can express u as a convolution of that function and v. So, we are looking to find

$$g(x,t) \text{ such that } \hat{g}(x,t) = e^{-\xi^2 t}.$$

We use the FIT:

$$g(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\xi^2 t} d\xi.$$

We can use some complex analysis to compute this integral. To do this, we shall complete the square in the exponent:

$$-\xi^2 t + ix\xi = -\left(\xi\sqrt{t} + \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}.$$

Therefore we are computing

$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} + \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right) d\xi.$$

Using a contour integral, we can in fact ignore the imaginary part. So, we compute (using a change of variables to $y = \xi\sqrt{t}$ so $t^{-1/2}dy = d\xi$)

$$\int_{\mathbb{R}} e^{-\xi^2 t} d\xi = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-y^2} dy = \frac{\sqrt{\pi}}{\sqrt{t}}.$$

Hence,

$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} + \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right) d\xi = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{x^2}{4t}}.$$

Recalling the factor of $1/(2\pi)$ we have

$$g(x,t) = \frac{1}{2\pi} \frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{x^2}{4t}} = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

Hence the solution is

$$u(x,t) = g * v(x) = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-(x-y)^2/(4t)} v(y) dy.$$

1.1.1. *Application to computing tricky integrals.* The following is a very useful observation:

$$\hat{f}(0) = \int_{\mathbb{R}} f(x) dx.$$

So, if you have the integral of a function, this is equal to the value of its Fourier transform at $\xi = 0$. So, if you can look up the Fourier transform of the function, like in Beta or Folland, then to compute the integral, no need for fancy contour integrals, simply pop $\xi = 0$ into the Fourier transform.

Here is an example:

$$\text{compute: } \int_{\mathbb{R}} \frac{1}{x^2 + 9} dx.$$

We see this is # 10 in Folland's TABLE 2. It is inevitably in BETA somewhere also... On the right side, we get the Fourier transform (with $a = 3$) is given by

$$\frac{\pi}{3} e^{-3|\xi|}.$$

So, this integral is the Fourier transform with $\xi = 0$, hence the value of the integral is

$$\frac{\pi}{3}.$$

That was pretty easy right? For something more complicated, you could have say

$$\int_{\mathbb{R}} f(x)g(x)dx,$$

with some icky functions f and g (see extra övning # 9). Now, you can use that the Fourier transform of a product is

$$(2\pi)^{-1}(\hat{f} * \hat{g})(\xi).$$

Hence, what you have above is

$$\int_{\mathbb{R}} f(x)g(x)dx = \int_{\mathbb{R}} e^{-i(0)x} f(x)g(x)dx = (2\pi)^{-1}(\hat{f} * \hat{g})(0).$$

So, if the Fourier transforms of these functions are somewhat better than the functions f and g, then the stuff on the right could be nicely computable and give you the integral on the left. Try # 9 to see how this works. (If you get stuck, Team Fourier is here to help! Just ask us!)

As another example, there is extra exercise number 10. It says you know the Fourier transform of $f(t)$ is $\frac{1}{|w|^3+1}$. We're supposed to compute

$$\int_{\mathbb{R}} |f * f'|^2 dt.$$

Yikes! Looks scary eh? Well, let's stay calm and carry on. We recognize an $\mathcal{L}^2$ norm looking thing. By the Plancharel theorem,

$$\int_{\mathbb{R}} |f * f'|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f * f'}|^2 dt.$$

Now we use the theorem on the properties of the Fourier transform which says

$$\widehat{f * f'}(\xi) = \hat{f}(\xi)\widehat{f'}(\xi).$$

Now we use that same theorem to say that

$$\widehat{f'}(\xi) = i\xi\hat{f}(\xi).$$

So, the stuff on the right is

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi) i\xi \hat{f}(\xi)|^2 d\xi.$$

We are given what the Fourier transform is, so we put it in there:

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\xi^2}{(|\xi|^3+1)^4} d\xi.$$

Now this isn't so terrible. It's an even function so this is

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{\xi^2}{(\xi^3+1)^4} d\xi.$$

It just so happens that the derivative of

$$\frac{1}{(\xi^3+1)^3} \text{ is } \frac{-9\xi^2}{(\xi^3+1)^4},$$

so

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi^2}{(\xi^3+1)^4} d\xi = \frac{-1}{9\pi} \left. \frac{1}{\xi^3+1}\right|_0^\infty = \frac{1}{9\pi}.$$

1.2. **Application to the inhomogeneous heat equation.** If you have an inhomogeneous IVP for the heat equation, here are two ways to deal with that:

(1) If the inhomogeneity is *time independent,* look for a steady state solution to solve the inhomogeneous equation. Then, solve the homogeneous equation, but change your initial data. If f is your steady state solution and v was your initial data (before f came along), solve the IVP for the homogeneous heat equation with IC $v - f$ rather than just v.
(2) If the inhomogeneity is *time dependent,* you can try to solve using the original method we did, that is by Fourier transforming the whole PDE.

Since we know how to do the first type of example, let us consider the second type of example. We want to solve an inhomogeneous heat equation on $\mathbb{R}$:

$$u_t - u_{xx} = G(x,t), \quad u(x,0) = v(x).$$

Let's try the Fourier transform method:

$$\partial_t \hat{u}(\xi,t) + \xi^2 \hat{u}(\xi,t) = \hat{G}(\xi,t).$$

This is a first order ODE. If you are a CHEMIST, then you did the special extra part of the course and actually learned how to solve this ODE in t. Pretty cool. To see how this works, treat ξ like a constant, and write

$$f'(t) + \xi^2 f(t) = \hat{G}(\xi,t).$$

The method says to first compute

$$e^{\int \xi^2 dt} = e^{\xi^2 t}.$$

Next compute

$$\int e^{\xi^2 t} \hat{G}(\xi,t) dt.$$

Then, the solution is

$$\frac{\int e^{\xi^2 t}\hat{G}(\xi,t)dt + C(\xi)}{e^{\xi^2 t}} = e^{-\xi^2 t}\int e^{\xi^2 s}\hat{G}(\xi,s)ds + C(\xi)e^{-\xi^2 t}.$$

Now, if we choose a primitive (anti-derivative) of $e^{\xi^2 s}\hat{G}(\xi,s)$ which is zero when $t = 0$, then we can simply set $C(\xi) = \hat{v}(\xi)$. So, to do this, we use the function

$$F(\xi,t) := \int_0^t e^{\xi^2 s}\hat{G}(\xi,s)ds.$$

By the Fundamental Theorem of Calculus, the t derivative of F is the integrand evaluated at t. There is too much t. Let me be more precise

$$\partial_t F(\xi,t)\big|_{t=t_0} = e^{\xi^2 t_0}\hat{G}(\xi,t_0).$$

That's what the FTC says. So, our solution as of now looks like

$$\hat{u}(\xi,t) = e^{-\xi^2 t}\int_0^t e^{\xi^2 s}\hat{G}(\xi,s)ds + \hat{v}(\xi)e^{-\xi^2 t}.$$

We need to figure out from whence this Fourier transform came (equivalently, invert the Fourier transform). This is a linear process, so we can deal with each piece separately and then add them. Well, the second part we did last time. We saw that the Fourier transform of

$$\frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y)dy$$

is

$$\hat{v}(\xi)e^{-\xi^2 t}.$$

Similarly, let's look at the first part. It is

$$e^{-\xi^2 t}\int_0^t e^{\xi^2 s}\hat{G}(\xi,s)ds = \int_0^t e^{-\xi^2(t-s)}\hat{G}(\xi,s)ds.$$

By the same calculations, the Fourier transform of

$$\frac{1}{2\sqrt{\pi(t-s)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} G(y,s)dy = e^{-\xi^2(t-s)}\hat{G}(\xi,s).$$

Yet again playing switch-a-roo with limits[1],

$$\mathcal{F}\left(\int_0^t \frac{1}{2\sqrt{\pi(t-s)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} G(y,s)dyds\right)(\xi) = \int_0^t e^{-\xi^2(t-s)}\hat{G}(\xi,s)ds.$$

Therefore, our full solution is

$$\int_0^t \frac{1}{2\sqrt{\pi(t-s)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} G(y,s)dyds + \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y)dy.$$

1.3. **The Sampling Theorem.**

Theorem 1. *Let $f \in L^2(\mathbb{R})$. We take the definition of the Fourier transform of f to be*

$$\int_{\mathbb{R}} e^{-ix\xi} f(x)dx,$$

and we then assume that there is $L > 0$ so that $\hat{f}(\xi) = 0\ \forall \xi \in \mathbb{R}$ with $|\xi| > L$. Then:

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{L}\right)\frac{\sin(n\pi - tL)}{n\pi - tL}.$$

Proof: This theorem is all about the interaction between Fourier series and Fourier coefficients and how to work with both simultaneously. Since the Fourier transform $\hat{f}$ has compact support, the following equality holds as elements of $L^2([-L,L])$,

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx.$$

We use the Fourier inversion theorem (FIT) to write

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt}\hat{f}(x)dx = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\hat{f}(x)dx.$$

On the left we have used the fact that $\hat{f}$ is supported in the interval $[-L,L]$, thus the integrand is zero outside of this interval, so we can throw that part of the integral away.

[1] Trust me!

We next substitute the Fourier expansion of $\hat{f}$ into this integral,

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} c_n e^{in\pi x/L} dx.$$

Let us take a closer look at the coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx = \frac{1}{2L} \int_{\mathbb{R}} e^{ix(-n\pi/L)} \hat{f}(x) dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

In the second equality we have used the fact that $\hat{f}(x) = 0$ for $|x| > L$, so by including that part we don't change the integral. In the third equality we have used the FIT!!! So, we now substitute this into our formula above for

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L} dx$$

This is approaching the form we wish to have in the theorem, but the argument of the function f has a pesky negative sign. That can be remedied by switching the order of summation, which does not change the sum, so

$$f(t) = \frac{1}{2L} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{-in\pi x/L} dx.$$

We may also interchange the summation with the integral[2]

$$f(t) = \frac{1}{2L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \int_{-L}^{L} e^{x(it - in\pi/L)} dx.$$

We then compute

$$\int_{-L}^{L} e^{x(it-in\pi/L)} dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)} \sin(Lt - n\pi).$$

Substituting,

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \frac{\sin(Lt - n\pi)}{Lt - n\pi}.$$

Of course my dyslexia has ended up with things being backwards, but it is not a problem because sine is odd so

$$\sin(Lt - n\pi) = -\sin(n\pi - Lt),$$

so

$$\frac{\sin(Lt - n\pi)}{Lt - n\pi} = \frac{-\sin(n\pi - Lt)}{Lt - n\pi} = \frac{\sin(n\pi - Lt}{n\pi - Lt}.$$

[2] None of this makes sense pointwise; we are working over L^2. The key property which allows interchange of limits, integrals, sums, derivatives, etc is *absolute convergence.* This is the case here because elements of L^2 have $\int |f|^2 < \infty$. That is precisely the type of absolute convergence required.

1.4. **Exercises for the week to be done oneself: hints.**

(1) (Eö 9) Compute (with help of Fourier transform)

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)} dx.$$

Hint: There are disguised zeros and ones hiding all over the place in mathematics. The above is equal to

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)} e^{-i(0)x} dx = \mathcal{F}\left(\frac{\sin x}{x}\frac{1}{x^2+1}\right)(0).$$

So, we now look at Table 2 in Folland, especially item number 8. It says that the Fourier transform of a product is a convolution of the Fourier transforms. So, we apply this to say

$$\mathcal{F}\left(\frac{\sin x}{x}\frac{1}{x^2+1}\right)(0) = \frac{1}{2\pi}\mathcal{F}\left(\frac{\sin x}{x}\right) * \mathcal{F}\left(\frac{1}{x^2+1}\right)(0).$$

Now we use items 10 and 13 from the same table, together with the definition of the convolution, to substitute for the Fourier transforms on the right side:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \pi\chi_1(0-y)\pi e^{-|y|} dy.$$

Recalling what χ_1 means:

$$= \frac{\pi}{2}\int_{-1}^{1} e^{-|y|} dy.$$

I leave it to you do compute the integral!

(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a, b) both directly and by using the known case for the interval $(-a, a)$.

Hint: Well, doing it directly we are computing

$$\int_a^b e^{-ix\xi} dx = \begin{cases} b-a & \xi = 0 \\ \frac{i}{\xi}\left(e^{-bi\xi} - e^{-ai\xi}\right) & \xi \neq 0 \end{cases}$$

To do it the other way, it's convenient to introduce some notations:

$$m := \frac{a+b}{2}, \ell := \frac{b-a}{2}.$$

Then our interval is $[m-\ell, m+\ell]$. So we are computing

$$\int_{m-\ell}^{m+\ell} e^{-ix\xi} dx.$$

To make this more familiar let's do a change of variables so that the integral goes from $-\ell$ to ℓ, so we let $t = x - m$, then $dt = dx$, so we are computing

$$\int_{-\ell}^{\ell} e^{-i(t+m)\xi} dt = e^{-im\xi}\int_{-\ell}^{\ell} e^{-it\xi} dt = e^{-im\xi}\hat{\chi}_{[-\ell,\ell]}(\xi).$$

So now for the Fourier transform of the characteristic function of the interval, that is the function $\chi_{[-\ell,\ell]}$ we can use the item 12 in Table 2 of Folland. With a little algebraic manipulations, one can show that these both roads lead to the same answer.

(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1+i\xi)^{-a}$ where Γ is the Gamma function.

Hint: one is computing

$$\int_0^\infty e^{-x}e^{-ix\xi}x^{a-1}dx = \int_0^\infty e^{-x(1+i\xi)}x^{a-1}dx.$$

On the other hand,

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt.$$

Try doing a substitution to relate these integrals...

(4) (7.2.12) For $a > 0$ let

$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$

Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.

Hint: The idea is basically repeated use of the items in Folland's Table 2, and using the FIT. First, compute the Fourier transform of $f_a * f_b$ which is $\hat{f}_a(\xi)\hat{f}_b(\xi)$, so you can write this stuff down. You will get something like $e^{-|x|\cdots}$. Next, use the FIT to return to $f_a * f_b$. Note that one way to write the FIT is

$$f(x) = \frac{1}{2\pi}\widehat{\hat{f}}(-x).$$

Do something similar for the second one...

(5) (Eö 6.d,e) Compute the Fourier transform of:

$$e^{-a|t|}\sin(bt), \quad (a, b > 0), \quad \frac{t}{t^2+2t+5}.$$

Hint: I might deal with the first one by splitting up the sine into its complex exponentials, using definition of Fourier transform, and just directly integrating. As for the second one, note that $t^2 + 2t + 5 = (t+1)^2 + 4$. Do a substitution in the definition of the Fourier transform, let $x = t + 1$. Then use item 10 on Folland's Table 2.

(6) (Eö 15) Find a solution to the equation

$$u(t) + \int_{-\infty}^t e^{\tau - t}u(\tau)d\tau = e^{-2|t|}.$$

Hint: This is a tricky one! First turn the integral into a convolution. How to do that? Try using $\Theta(\tau)e^{-|\tau|}$. Write out the convolution of that function together with $u(\tau)$. Next, Fourier transform both sides of the equation. So you will get

$$\hat{u}(\xi) + (\widehat{\Theta(\tau)e^{-|\tau|}})(\xi)\hat{u}(\xi) = \widehat{e^{-2|t|}}(\xi).$$

Compute the Fourier transforms of everything except u. Solve the equation for $\hat{u}(\xi)$. Then use the FIT. When you use the FIT, if you do it using contour integrals and the residue, you will need to think about the cases $x > 0$ and $x < 0$ separately. For $x > 0$ the up-rainbow will work. For $x < 0$ the down-rainbow will work.

(7) (Eö 11) For the function

$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w} e^{iwt} dw,$$

compute

$$\int_{\mathbb{R}} f(t)\cos(t)dt, \quad \int_{\mathbb{R}} |f(t)|^2 dt.$$

Hint: This is tricky also. Let me define a new function for us:

$$\phi(w) := \chi_{[0,2]}(w) \frac{\sqrt{2}}{1+w}.$$

Then

$$f(t) = \widehat{\phi}(-t).$$

Oh no she didn't. Yeah. So, for the first one, note that this integral is, expanding the cosine as a sum of complex exponentials

$$\int_{\mathbb{R}} f(t)\cos(t)dt = \frac{1}{2}\left(\hat{f}(1) + \hat{f}(-1)\right).$$

Play around with the FIT and the fact that $f(t) = \widehat{\phi}(-t)$ to figure out the right side. Next, note that

$$\int_{\mathbb{R}} |f(t)|^2 dt = \int_{\mathbb{R}} |\widehat{\phi}(-t)|^2 dt = 2\pi \int_{\mathbb{R}} |\phi(t)|^2 dt.$$

The integral of $|\phi|^2$ is hopefully not that terrible...

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).