FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Today we shall investigate some transforms related to the Fourier transform. The first two can be used to solve PDEs on half lines, *if the boundary condition* is suitable.

Definition 1. *Let f be in $\mathcal{L}^1$ or $\mathcal{L}^2$ on $(0,\infty)$. The Fourier cosine transform,*

$$\mathcal{F}_c(f)(\xi) := \int_0^\infty f(x)\cos(\xi x)dx.$$

The Fourier sine transform,

$$\mathcal{F}_s(f)(\xi) := \int_0^\infty f(x)\sin(\xi x)dx.$$

As with the Fourier transform, the Fourier sine and cosine transforms also have inversion formula.

Theorem 2. *Assume that $f \in \mathcal{L}^2[0,\infty)$. Then we have the Fourier cosine inversion formula*

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(f)(\xi)\cos(x\xi)d\xi.$$

We also have the Fourier sine inversion formula

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi.$$

Proof: First, let us extend f evenly to $\mathbb{R}$, denoting this extension by f_e, so that $f_e(-x) = f_e(x)$. We compute the standard Fourier transform:

$$\hat{f}_e(\xi) = \int_{\mathbb{R}} f_e(x)e^{-ix\xi}dx = \int_{\mathbb{R}} f_e(x)(\cos(x\xi) - i\sin(x\xi))dx = 2\int_0^\infty f(x)\cos(x\xi)dx.$$

The term with the sine has dropped out because $f_e(x)\sin(x\xi)$ is an odd function of x. The term with the cosine gets doubled because $f_e(x)\cos(x\xi)$ is an even function. So, all together we have computed:

$$\hat{f}_e(\xi) = 2\int_0^\infty f(x)\cos(x\xi)dx = 2\mathcal{F}_c(f)(\xi).$$

Since the cosine is an even function,

$$\hat{f}_e(\xi) = \hat{f}_e(-\xi).$$

So, we also have that $\mathcal{F}_c(f)$ is an even function. The inversion formula (FIT) says that

$$f_e(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}_e(\xi)d\xi = \frac{1}{\pi}\int_{\mathbb{R}} e^{ix\xi}\mathcal{F}_c(f)(\xi)d\xi$$

$$= \frac{1}{\pi}\int_{\mathbb{R}}(\cos(x\xi) + i\sin(x\xi))\mathcal{F}_c(f)(\xi)d\xi = \frac{2}{\pi}\int_0^\infty e^{ix\xi}\mathcal{F}_c(f)(\xi)d\xi.$$

This is the cosine-FIT! Above we have used the fact that $\mathcal{F}_c(f)$ is an even function. Hence its product with the cosine is also an even function, so that part of the integral gets a factor of two when we integrate only over the positive real line. The product of an even function like $\mathcal{F}_c(f)$ with an odd function, like the sine, is odd, so that integral vanishes.

On the other hand, we may also define the odd extension, f_o which satisfies $f_o(-x) = -f_o(x)$ (for $x \neq 0$). The value of f at zero is not really important at this moment.[1] We compute the standard Fourier transform of the odd extension:

$$\hat{f}_o(\xi) = \int_{\mathbb{R}} f_o(x)e^{-ix\xi}dx = \int_{\mathbb{R}} f_o(x)(\cos(x\xi) - i\sin(x\xi))dx = -2i\int_0^\infty f(x)\sin(x\xi)dx$$

$$= -2i\mathcal{F}_s(f)(\xi).$$

Above, we have used the fact that f_o is odd, and therefore so is its product with the cosine. On the other hand, the product with the sine is an even function, which explains the factor of 2. Since the sine itself is odd, we have that $\hat{f}_o$ is an odd function and similarly $\mathcal{F}_s(f)(\xi)$ is also an odd function. We apply the FIT:

$$f_o(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}_o(\xi)d\xi = -\frac{i}{\pi}\int_{\mathbb{R}}(\cos(x\xi) + i\sin(x\xi))\mathcal{F}_s(f)(\xi)d\xi$$

$$= \frac{1}{\pi}\int_{\mathbb{R}}\sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi = \frac{2}{\pi}\int_0^\infty \sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi).$$

This is the sine-FIT! Above we have used the fact that $\mathcal{F}_s(f)$ is an odd function, and therefore so is its product with the cosine. On the other hand the product of two odd functions is an even function, so that is the reason for the factor of 2.

1.1. **Heat equation on a semi-infinite rod with insulated end.** We have found ourselves in possession of a giant rod which is insulated at the one end and goes out to infinity at the other. Pretty neat. It has an initial temperature distribution given by a function $f(x)$. We therefore wish to solve the problem:

$$u_t - u_{xx} = 0, \quad u_x(0,t) = 0, \quad u(x,0) = f(x), \quad x \in [0,\infty).$$

Assume that by some method, we have obtained a solution $u(x,t)$ defined on $[0,\infty)_x \times [0,\infty)_t$. To see if we may use a Fourier sine or cosine transform method,

[1]This is because we are working in $\mathcal{L}^2$ which ignores sets of measure zero, and a single point is a set of measure zero.

let us see what happens when we extend our solution evenly or oddly. The even extension would satisfy, by the cosine-FIT:

$$u_e(x,t) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(u)(\xi)\cos(x\xi)d\xi.$$

The odd extension would satisfy, by the sine-FIT

$$u_o(x,t) = \frac{2}{\pi}\int_0^\infty \sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi.$$

OBS! The extension matches up with our original function on the positive real line (that is how an extension works!)

We need the derivative with respect to x to vanish at $x = 0$. Let's just differentiate these expressions. Note that the x dependence is only in the sine or cosine term so we have:

$$\partial_x u_e(x,t) = -\frac{2}{\pi}\int_0^\infty \mathcal{F}_c(u)(\xi)\xi\sin(x\xi)d\xi \implies \partial_x u_e(0,t) = 0.$$

On the other hand

$$\partial_x u_o(x,t) = \frac{2}{\pi}\int_0^\infty \xi\cos(x\xi)\mathcal{F}_s(u)(\xi)d\xi \implies \partial_x u_o(0,t) = \frac{2}{\pi}\int_0^\infty \xi\mathcal{F}_s(u)(\xi)d\xi = ???$$

The even extension automatically gives us the desired boundary condition whereas the odd extension leads to something complicated looking, which we have no reason to know is zero.

This indicates that we have a decent chance of solving the problem by:

(1) Extend the initial data *evenly* to the real line.
(2) Solve the problem using the Fourier transform on the real line.
(3) Verify that the solution satisfies all the conditions: the PDE, the IC, and the BC.

We do this. Extend f evenly, and write the extension as f_e. Then the solution to the homogeneous heat equation on the real line with initial data f_e is

$$u_e(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} f_e(y)e^{-\frac{(x-y)^2}{4t}}dy.$$

We split up the integral:

$$\int_{-\infty}^0 f_e(y)e^{-(x-y)^2/(4t)}dy + \int_0^\infty f_e(y)e^{-(x-y)^2/(4t)}dy$$

$$= -\int_\infty^0 f_e(z)e^{-(z+x)^2/(4t)}dz + \int_0^\infty f_e(y)e^{-(x-y)^2/(4t)}dy.$$

Above we made the substitution that $z = -y$ in the first integral. Due to the evenness of f_e, nothing happens when we change $y = -z$. Reversing the limits of integration we obtain

$$-\int_\infty^0 f_e(z)e^{-(z+x)^2/(4t)}dz = \int_0^\infty f_e(z)e^{-(z+x)^2/(4t)}dz = \int_0^\infty f_e(y)e^{-(x+y)^2/(4t)}dy.$$

So, all together we have

$$u_e(x,t) = \frac{1}{2\sqrt{\pi t}}\int_0^\infty f(y)\left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\right)dy.$$

Is this an even function? Let us verify:

$$e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} = e^{-\frac{(-x-y)^2}{4t}} + e^{-\frac{(-x+y)^2}{4t}}.$$

AWESOME! Our solution to the heat equation in this way is EVEN. Therefore, it is the same on the left and right sides. So, we can simply let

$$u(x,t) = u_e(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty f(y) \left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

The way we have built it, it satisfies the IC, BC, and the PDE!

Exercise 1. *Assume that f is also continuous. Use the convolution approximation theorem to prove that $u(x,t) \to f(x)$ as $t \to 0$.*

1.2. **Discrete and fast Fourier transform.** We have seen that computing the Fourier transform is not the easiest thing in the world. The example with the Gaussian involving all those tricks: completing the square, complex analysis and contour integral is a nice and easy case. However, in the *real world* you may come across functions and not know how to compute the Fourier transform by hand, nor be able to find it in BETA. It could be lurking in one of our giant handbooks of calculations (Abramowitz & Stegun, Gradshteyn & Rhizik, to name a few). Or it could simply never have been computed analytically. In this case you may compute something called the *discrete Fourier transform.*

We start with a function, $f(t)$, and think of analyzing $f(t)$ as *time analysis,* whereas analyzing $\hat{f}(\xi)$ as *frequency analysis.* We shall consider a finite dimensional Hilbert space:

$$\mathbb{C}^N = \left\{ (s_n)_{n=0}^{N-1}, \quad s_n \in \mathbb{C}, \quad \langle (s_n), (t_n) \rangle := \sum_{n=0}^{N-1} s_n \overline{t_n} \right\}.$$

Now let

$$e_k(n) := \frac{e^{2\pi i k n/N}}{\sqrt{N}}.$$

Proposition 3. *Let*

$$e_k := (e_k(n))_{n=0}^{N-1}.$$

Then

$$\{e_k\}_{k=0}^{N-1}$$

are an ONB of $\mathbb{C}^N$.

Proof: We simply compute. It is so cute and discrete!

$$\langle e_k, e_j \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k n/N} e^{-2\pi i j n/N} = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i (k-j) n/N}.$$

If $j = k$ the terms are all 1, and so the total is N which divided by N gives 1. Otherwise, we may without loss of generality assume that $k > j$ (swap names if not the case). Then we are staring at a geometric series! We know how to sum it

$$\sum_{n=0}^{N-1} e^{2\pi i (k-j) n/N} = \frac{1 - e^{2\pi i (k-j) N/N}}{1 - e^{2\pi i (k-j)/N}} = 0.$$

Here it is super important that $k-j$ is a number between 1 and $N-1$. We know this because $0 \leq j < k \leq N-1$. Hence when we subtract j from k, we get something between 1 and $N-1$. So we are not dividing by zero!

Now we shall fix T small and N large and look at $f(t)$ just on the interval $[0,(N-1)T]$. Let

$$f(n) := f(t_n) := f(nT), \quad t_n = nT.$$

Basically, we're going to identify f with an element of $\mathbb{C}^N$, namely

$$(f(n))_{n=0}^{N-1}.$$

Definition 4. *The discrete Fourier transform is for*

$$w_k := \frac{2\pi k}{NT}$$

defined to be

$$F_k = F(w_k) := \langle (f(n)), e_k \rangle = \sum_{n=0}^{N-1} \frac{f(t_n) e^{-2\pi i k n/N}}{\sqrt{N}}.$$

This can also be written as

$$\sum_{n=0}^{N-1} \frac{f(t_n) e^{-i w_k t_n}}{\sqrt{N}}.$$

Proposition 5. *We have the inversion formula*

$$f(t_n) = \sum_{k=0}^{N-1} F(w_k) e_n(k) = \langle (F_k), \overline{e}_n \rangle.$$

Proof: We simply compute this stuff. By definition

$$\langle (F_k), \overline{e}_n \rangle = \sum_{k=0}^{N-1} F(w_k) e_n(k),$$

because taking two conjugates gives us back the original guy. Now, we insert the definition of $F(w_k)$ which gives us another sum, so we use a different index there. Hence we have

$$\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \frac{f(t_m) e^{-i w_k t_m}}{\sqrt{N}} \frac{e^{2\pi i k n/N}}{\sqrt{N}} = \frac{1}{N} \sum \sum f(t_m) e^{-2\pi i k m/N} e^{2\pi i k n/N}$$

$$= \frac{1}{N} \sum \sum f(t_m) e^{2\pi i k (n-m)/N} = \frac{1}{N} \sum_{m=0}^{N-1} f(t_m) \sum_{k=0}^{N-1} e^{2\pi i k (n-m)/N}$$

$$= \sum_{m=0}^{N-1} f(t_m) \sum_{k=0}^{N-1} \frac{e^{-2\pi i k m/N}}{\sqrt{N}} \frac{\overline{e^{-2\pi i k n/N}}}{\sqrt{N}} = \sum_{m=0}^{N-1} f(t_m) \langle e_m, e_n \rangle.$$

By the proposition we just proved before,

$$\langle e_m, e_n \rangle = \delta_{n,m} = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases}$$

So, the only term which survives is when $m = n$, and so we get

$$f(t_n).$$

Now, we can see this as a sort of matrix multiplication. To compute the full frequency Fourier transform vector, we should compute

$$\begin{bmatrix} F(w_0) \\ F(w_1) \\ \dots \\ F(w_{N-1}) \end{bmatrix}.$$

This is given by the product of the matrix

$$\begin{bmatrix} \bar{e}_0 & \bar{e}_1 & \dots \bar{e}_{N-1} \end{bmatrix}$$

whose columns are

$$\bar{e}_n = \begin{bmatrix} e^0 \\ e^{-2\pi i n/N} \\ e^{-2\pi i(2)n/N} \\ \dots e^{-2\pi i k n/N} \\ \dots \\ e^{-2\pi i n(N-1)/N} \end{bmatrix}$$

together with the vector

$$\begin{bmatrix} f(t_0) \\ f(t_1) \\ \dots \\ f(t_{N-1}) \end{bmatrix}$$

That is

$$\begin{bmatrix} F(w_0) \\ F(w_1) \\ \dots \\ F(w_{N-1}) \end{bmatrix} = \begin{bmatrix} \bar{e}_0 & \bar{e}_1 & \dots \bar{e}_{N-1} \end{bmatrix} \begin{bmatrix} f(t_0) \\ f(t_1) \\ \dots \\ f(t_{N-1}) \end{bmatrix}$$

This is a LOT of calculations. We can speed it up by being clever. Many calculations are repeated in fact. Assume that $N = 2^X$ for some giant power X. The idea is to split up into even and odd terms. We do this:

$$F(w_k) = \frac{1}{\sqrt{N}} \left[\sum_{j=0}^{\frac{N}{2}-1} f(t_{2j}) e^{-2\pi i k(2j)/N} + \sum_{j=0}^{\frac{N}{2}-1} f(t_{2j+1}) e^{-2\pi i k(2j+1)/N} \right].$$

We introduce the slightly cumbersome notation:

$$e_N^k(n) = e^{-2\pi i k n/N}.$$

Then,

$$e_N^k(2j) = e^{-2\pi i k(2j)/N} = e^{-2\pi i k j/(N/2)} = e_{N/2}^k(j).$$

Now we only need an $\frac{N}{2} \times \frac{N}{2}$ matrix! You see, writing this way,

$$F(w_k) = \frac{1}{\sqrt{N}} \left[\sum_{j=0}^{\frac{N}{2}-1} f(t_{2j}) e_{N/2}^k(j) + e_N^k(1) \sum_{j=0}^{\frac{N}{2}-1} f(t_{2j+1}) e_{N/2}^k(j) \right].$$

We can repeat this many times because N is a power of 2. We just keep chopping in half. If we do this as many times as possible, we will need to do on the order of

$$\frac{N}{2} \log_2(N)$$

computations. This is in comparison to the original method which had an $N \times N$ matrix and was thus on the order of N^2 computations. For example, if $N = 2^{10}$, then comparing $N^2 = 2^{20}$ to $\frac{N}{2}\log_2 N = 2^9 * 10$, we see that

$$\frac{2^{10} * 5}{2^{20}} = \frac{x}{100} \implies 100 * 2^{10} * 5 = 2^{20}x \implies 2^2 * 5^3 * 2^{10}2^{-20} = x,$$

so

$$5^3 2^{-8} = x \approx 0.488.$$

This means the amount of work we are doing by using the FFT is less than 0.5% of the work done using the standard DFT. In other words, we save over 99.5% by doing the FFT. That's why it's called FAST.

1.3. **Answers to the exercises to be done oneself.**

(1) (Eö 9) Compute (with help of Fourier transform)

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)} dx.$$

(This is in the back of the EÖ document!)

(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a, b) both directly and by using the known case for the interval $(-a, a)$. (This is in the back of the EÖ document!)

(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1 + i\xi)^{-a}$ where Γ is the Gamma function.

Well, there are not really answers to make sense of here. My hint was to do a substitution of variables:

$$\hat{f}(\xi) = \int_0^\infty e^{-ix\xi - x} x^{a-1} dx.$$

On the other hand

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt.$$

So let's try making

$$x(1 + i\xi) = t \implies dx(1 + i\xi) = dt \implies \frac{dt}{1 + i\xi} = dx.$$

Our integral becomes

$$\hat{f}(\xi) = \int_0^{(1+i\xi)\infty} e^{-t} \left(\frac{t}{1+i\xi}\right)^{a-1} \frac{dt}{1+i\xi}$$
$$= (1 + i\xi)^{-a} \int_0^{(1+i\xi)\infty} e^{-t} t^{a-1} dt.$$

What is up with those weird limits of integration? Let's investigate by drawing a picture.

Integrate along the line from 0 to $(1 + i\xi)R = R + iR\xi$. For $\xi > 0$ that is the first diagonal bit. Next, integrate from $R + iR\xi$ to R. The integrate back along the real axis from R to zero. Our integrand is $e^{-z}z^{a-1}$. Inside the triangle it's holomorphic. So by complex analysis the integral around the triangle is zero. Since $|e^{-z}| = e^{-x}$ if $z = x + iy$ for $x, y \in \mathbb{R}$, along the right side of the triangle the integral is super small, tending to zero. That says the the integral along this funny diagonal line and

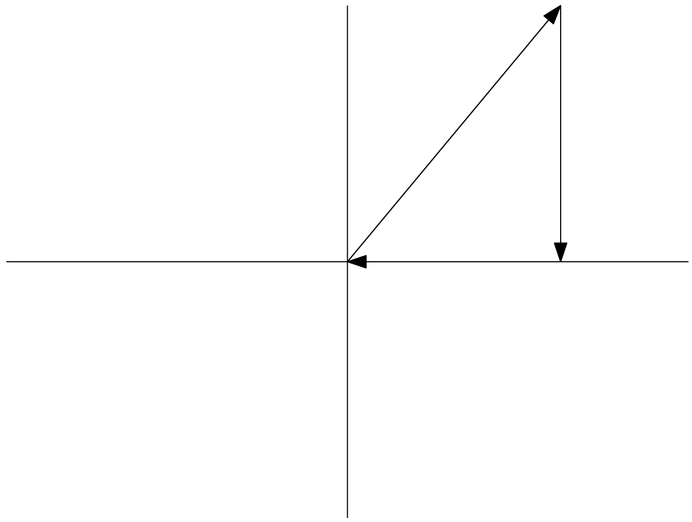

FIGURE 1. The contour integral for the exercise about the gamma function.

the integral going from R to 0 are tending to be equal. More precisely $\lim_{R\to\infty}\int_0^{R(1+i\xi)} f(z)dz + \int_R^0 f(z)dz = 0$. Hence since flipping the integral changes its sign $\lim_{R\to\infty}\int_0^{R(1+i\xi)} f(z)dz = \int_0^\infty f(z)dz$. So

$$\hat{f}(\xi) = (1+i\xi)^{-a}\int_0^{(1+i\xi)\infty} e^{-t}t^{a-1}dt = (1+i\xi)^{-a}\int_0^\infty e^{-t}t^{a-1}dt.$$

This is $(1+i\xi)^{-a}\Gamma(a)$.

(4) (7.2.12) For $a > 0$ let

$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$

Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.

So we transform:

$$\widehat{f_a * f_b}(\xi) = \widehat{f_a}(\xi)\widehat{f_b}(\xi) = e^{-a|\xi|-b|\xi|} = e^{-(a+b)|\xi}.$$

Now we use the FIT to say:

$$f_a * f_b(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(a+b)|\xi|}e^{ix\xi}d\xi.$$

OBS! The integral on the right side this is the Fourier transform of $e^{-(a+b)|\xi|}$ at the point $-x$ rather than x. So we use our beloved Table 2 (item 11) to say that the Fourier transform of this function at the point $-x$ is

$$2(a+b)(x^2+(a+b)^2)^{-1},$$

so substituting

$$f_a * f_b(x) = \frac{1}{2\pi}2(a+b)(x^2+(a+b)^2)^{-1} = \frac{(a+b)}{\pi(x^2+(a+b)^2)} = f_{a+b}(x).$$

We do the same trick to solve the g exercise, yo.

$$\widehat{g_a * g_b}(\xi) = \widehat{g_a}(\xi)\widehat{g_b}(\xi) = \chi_a(\xi)\chi_b(\xi) = \chi_{\min(a,b)}(\xi).$$

The last step follows from the the definition of the characteristic function. So, we use the FIT again to say:

$$g_a * g_b(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\chi_{\min(a,b)}(\xi)d\xi.$$

Same trick: integral on the right is the Fourier transform of $\chi_{\min(a,b)}$ at the point $-x$ (rather than x). So we use our favorite Table 2 to say that

$$g_a * g_b(x) = \frac{1}{2\pi}x^{-1}2\sin(\min(a,b)x) = \frac{\sin(\min(a,b)x)}{\pi x} = g_{\min(a,b)}(x).$$

(5) (Eö 6.d,e) Compute the Fourier transform of:

$$e^{-a|t|}\sin(bt), \quad (a,b>0), \quad \frac{t}{t^2+2t+5}.$$

(This is in the back of the EÖ document!)

(6) (Eö 15) Find a solution to the equation

$$u(t) + \int_{-\infty}^{t} e^{\tau-t}u(\tau)d\tau = e^{-2|t|}.$$

(This is in the back of the EÖ document!)

(7) (Eö 11) For the function

$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w}e^{iwt}dw,$$

compute

$$\int_{\mathbb{R}} f(t)\cos(t)dt, \quad \int_{\mathbb{R}} |f(t)|^2 dt.$$

(This is in the back of the EÖ document!)

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).