

FOURIER ANALYSIS & METHODS

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications*, by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Let's do another example involving a half-space problem, in which we will use the Fourier sine and cosine transforms as guiding intuition.

1.1. Dirichlet problem in a quadrant. Consider the problem

$$u_{xx} + u_{yy} = 0, \quad x, y > 0, \quad u(x, 0) = f(x), \quad u(0, y) = g(y).$$

To deal with these inhomogeneities let us instead solve two nicer problems:

- (1) $w_{xx} + w_{yy} = 0, \quad x, y > 0, \quad w(x, 0) = f(x), \quad w(0, y) = 0.$
- (2) $v_{xx} + v_{yy} = 0, \quad x, y > 0, \quad v(x, 0) = 0, \quad v(0, y) = g(y).$

Let us take the Fourier transform in the x variable. We obtain:

$$-\xi^2 \hat{w}(\xi, y) + \partial_{yy} \hat{w}(\xi, y) = 0 \implies \hat{w}(\xi, y) = A(\xi) e^{-|\xi|y} + B(\xi) e^{|\xi|y}.$$

The functions A and B can depend on ξ but not on y . We would like to use Fourier methods which requires staying within \mathcal{L}^2 . Hence we do not want the second solution because $y > 0$ so it is very much not in \mathcal{L}^2 . Thus, we keep the first solution. The boundary condition then says that we need

$$\hat{w}(\xi, 0) = A(\xi) = \hat{f}(\xi) \implies \hat{w}(\xi, y) = \hat{f}(\xi) e^{-|\xi|y}.$$

We look at table 2 of Folland to find a function whose Fourier transform is $e^{-|\xi|y}$. OBS! The transform is occurring in the x variable, from whose perspective y is a constant. Thus, the item on the table is a slight modification of 10, in particular the function

$$\frac{y}{\pi} (x^2 + y^2)^{-1} \text{ has Fourier transform in the } x \text{ variable } e^{-y|\xi|}.$$

So, we are basically looking for a convolution of f with this function on the left side. However, we are only working on the positive quadrant, a half space type problem. Moreover, we have the boundary condition at $w(0, y) = 0$. The idea is to use either an even or odd extension of f . To get the condition $w(0, y) = 0$, which is the Dirichlet boundary condition, we should use the odd extension of f , that is f_o . The intuition behind this is that if we think about the Fourier sine and cosine

transforms, it is the sine itself which vanishes at zero. It is also the Fourier sine transform which corresponds to odd functions. So, this part of the solution

$$w(x, y) = \int_{-\infty}^{\infty} f_o(z) \frac{y}{\pi((x-z)^2 + y^2)} dz = \int_{-\infty}^0 f_o(z) \frac{y}{\pi((x-z)^2 + y^2)} dz + \int_0^{\infty} f(z) \frac{y}{\pi((x-z)^2 + y^2)} dz$$

We do a substitution in the first integral, with $t = -z$

$$\begin{aligned} &= \int_{-\infty}^0 f_o(z) \frac{y}{\pi((x-z)^2 + y^2)} dz = - \int_{\infty}^0 f_o(-t) \frac{y}{\pi((x+t)^2 + y^2)} dt \\ &= \int_0^{\infty} f(t) \frac{y}{\pi((x+t)^2 + y^2)} dt = - \int_0^{\infty} f(t) \frac{y}{\pi((x+t)^2 + y^2)} dt. \end{aligned}$$

Re-naming the variable of integration z , we get

$$w(x, y) = \int_0^{\infty} f(z) \left[\frac{y}{\pi((x-z)^2 + y^2)} - \frac{y}{\pi((x+z)^2 + y^2)} \right] dz.$$

The other problem is basically identical, we simply Fourier transform in the y variable. Thus the solution to the second problem is

$$v(x, y) = \int_0^{\infty} g(z) \left[\frac{x}{\pi((y-z)^2 + x^2)} - \frac{x}{\pi((y+z)^2 + x^2)} \right] dz.$$

We obtain the full solution by adding:

$$u(x, y) = w(x, y) + v(x, y).$$

1.2. The Laplace transform. We shall now enter Chapter 8, and learn about another useful transform, known as the Laplace transform.

Definition 1. Assume that

$$\boxed{\text{lap0}} \quad (1.1) \quad f(t) = 0 \quad \forall t < 0,$$

and that there exists $a, C > 0$ such that

$$\boxed{\text{lapa}} \quad (1.2) \quad |f(t)| \leq C e^{at} \quad \forall t \geq 0.$$

Then for we define for $z \in \mathbb{C}$ with $\Re(z) > a$ the Laplace transform of f at the point z to be

$$\mathfrak{L}f(z) = \hat{f}(-iz) = \int_0^{\infty} f(t) e^{-zt} dt.$$

Let us verify that this is well defined. To do so, we estimate

$$\begin{aligned} |\mathfrak{L}f(z)| &\leq \int_0^{\infty} |f(t) e^{-zt}| dt \leq \int_0^{\infty} C e^{at} |e^{-zt}| dt = \int_0^{\infty} e^{at} e^{-\Re(z)t} dt \\ &= \left. \frac{e^{t(a-\Re(z))}}{a-\Re(z)} \right|_0^{\infty} = \frac{1}{\Re(z)-a}. \end{aligned}$$

Above we have used the fact that

$$|e^{\text{complex number}}| = e^{\text{real part}}.$$

Due to this beautiful convergence, $\mathfrak{L}f(z)$ is holomorphic in the half plane $\Re(z) > a$. This is because we may differentiate under the integral sign due to the absolute convergence of the integral. The assumption that $f(t) = 0$ for all negative t is

not actually necessary, we could just make it so. For this purpose we define the *heavyside* function, commonly denoted by

$$\Theta(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

If we have some f defined on \mathbb{R} which satisfies \Lap0 but is not \Lap1 , we can apply the Laplace transform to Θf . Another thing which can happen is that we have a function which is only defined on $[0, \infty)$. In that case, we can just extend it to be identically zero on $(-\infty, 0)$.

Let's familiarize ourselves with the Laplace transform by demonstrating some of its fundamental properties.

Proposition 2 (Properties of \mathfrak{L}). *Assume f and g satisfy \Lap2 and \Lap0 , then*

- (1) $\mathfrak{L}f(x + iy) \rightarrow 0$ as $|y| \rightarrow \infty$ for all $x > a$.
- (2) $\mathfrak{L}f(x + iy) \rightarrow 0$ as $x \rightarrow \infty$ for all y .
- (3) $\mathfrak{L}(\Theta(t - a)f(t - a))(z) = e^{-az}\mathfrak{L}f(z)$.
- (4) $\mathfrak{L}(e^{ct}f(t))(z) = \mathfrak{L}f(z - c)$.
- (5) $\mathfrak{L}(f(at)) = a^{-1}\mathfrak{L}f(a^{-1}z)$.
- (6) *** If f is continuous and piecewise \mathcal{C}^1 on $[0, \infty)$, and f' satisfies \Lap2 and \Lap0 , then

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

- (7) $\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}f(z)$.
- (8) $\mathfrak{L}(tf(t))(z) = -(\mathfrak{L}f)'(z)$.
- (9) $\mathfrak{L}(f * g)(z) = \mathfrak{L}f(z)\mathfrak{L}g(z)$.

Proof: There's a bunch of stars next to #6 because it's the reason the Laplace transform is useful for solving PDEs and ODEs. It's quite similar to how the Fourier transform takes in derivatives and spits out multiplication. Intuitively, this fact about \mathfrak{L} should jive with the similar fact about \mathcal{F} because well, the Laplace transform is just the Fourier transform taken at a complex point.

The first statement

$$\mathfrak{L}f(z) = \int_0^\infty e^{-(x+iy)t} f(t) dt = \int_0^\infty e^{-xt} f(t) e^{-iyt} dt = \hat{g}(y),$$

for the function

$$g(t) = e^{-xt} f(t).$$

The Riemann-Lebesgue lemma says that $\hat{g}(y) \rightarrow 0$ when $|y| \rightarrow \infty$. The second statement is more satisfying because we just compute and estimate directly. We did this estimate above already, where we got

$$|\mathfrak{L}f(z)| \leq \frac{1}{\Re(z) - a} \rightarrow \infty \text{ when } \Re(z) = x \rightarrow \infty.$$

The third one is also a direct computation:

$$\mathfrak{L}(\Theta(t - a)f(t - a))(z) = \int_0^\infty \Theta(t - a)f(t - a)e^{-zt} dt = \int_{-a}^\infty \Theta(s)f(s)e^{-z(s+a)} ds.$$

Above we did the substitution $s = t - a$ so $ds = dt$. Since f and the Heavyside function are zero for negative s , and the Heavyside function is 1 for positive s , this is

$$e^{-za} \int_0^\infty f(s)e^{-zs} ds = e^{-za} \mathfrak{L}f(z).$$

Similarly, we directly compute

$$\mathfrak{L}(e^{ct}f)(z) = \int_0^\infty e^{ct}e^{-zt}f(t)dt = \int_0^\infty e^{-(z-c)t}f(t)dt = \mathfrak{L}f(z-c).$$

Again no surprise, we compute

$$\mathfrak{L}(f(at))(z) = \int_0^\infty e^{-zt}f(at)dt = \int_0^\infty e^{-zs/a}f(s)\frac{ds}{a} = a^{-1}\mathfrak{L}f(z/a).$$

Here we used the substitution $s = at$ so $a^{-1}ds = dt$. Now we are finally getting to the important one:

$$\mathfrak{L}(f')(z) = \int_0^\infty e^{-zt}f'(t)dt = e^{-zt}f(t)|_0^\infty + \int_0^\infty ze^{-zt}f(t)dt.$$

We have used integration by parts above. By [\(1.2\)](#) and since $\Re(z) > a$, the limit as $t \rightarrow \infty$ is zero, and so we get

$$\mathfrak{L}(f')(z) = -f(0) + z\mathfrak{L}f(z).$$

Awesome. Next we define

$$F(t) = \int_0^t f(s)ds.$$

Then, we use the preceding fact:

$$\mathfrak{L}(F')(z) = z\mathfrak{L}F(z) - F(0) = z\mathfrak{L}F(z).$$

Since $F' = f$ we get

$$z^{-1}\mathfrak{L}(f)(z) = \mathfrak{L}\left(\int_0^t f(s)ds\right)(z).$$

Next, we compute:

$$\begin{aligned} \mathfrak{L}(tf(t))(z) &= \int_0^\infty te^{-zt}f(t)dt = \int_0^\infty \frac{d}{dz}(-e^{-zt})f(t)dt \\ &= \frac{d}{dz}\left(-\int_0^\infty e^{-zt}f(t)dt\right) = -(\mathfrak{L}f)'(z). \end{aligned}$$

Yes, we have used the absolute convergence of the integral to swap limits. It's legit yo. Finally,

$$\mathfrak{L}(f * g)(z) = \mathcal{F}(f * g)(-iz) = \hat{f}(-iz)\hat{g}(-iz) = \mathfrak{L}f(z)\mathfrak{L}g(z).$$

☪

We may also use the notation

$$\tilde{f}(z) = \mathfrak{L}f(z).$$

1.3. Applications to solving PDEs and ODEs. We see that

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Let's do it again:

$$\mathfrak{L}(f'')(z) = z\mathfrak{L}(f')(z) - f'(0) = z(z\mathfrak{L}f(z) - f(0)) - f'(0) = z^2\mathfrak{L}f(z) - zf(0) - f'(0).$$

In general:

Proposition 3. *Assume that everything is defined, then*

$$\mathfrak{L}(f^{(k)})(z) = z^k \mathfrak{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0)z^{j-1}.$$

Proof: Well clearly we should do a proof by induction! Check the base case first:

$$\mathfrak{L}(f')(z) = z \mathfrak{L}f(z) - f(0).$$

Here $k = 1$ and the sum has only one term with $j = k = 1$. It works. Now we assume the above formula holds and we show it for $k + 1$. We compute

$$\mathfrak{L}(f^{(k+1)})(z) = \mathfrak{L}((f^{(k)})')(z) = z \mathfrak{L}(f^{(k)})(z) - f^{(k)}(0).$$

By induction this is

$$z \left(z^k \mathfrak{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0)z^{j-1} \right) - f^{(k)}(0).$$

This is

$$z^{k+1} \mathfrak{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0)z^j - f^{(k)}(0).$$

Let us change our sum: let $j + 1 = l$. Then our sum is

$$\sum_{l=2}^{k+1} f^{k-(l-1)}(0)z^{l-1} = \sum_{l=2}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

Observe that

$$f^{(k)}(0) = f^{k+1-1}(0)z^{1-1}.$$

Hence

$$- \sum_{j=1}^k f^{(k-j)}(0)z^j - f^{(k)}(0) = - \sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

So, we have computed

$$\mathfrak{L}(f^{(k+1)})(z) = z^{k+1} \mathfrak{L}f(z) - \sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

That is the formula for $k + 1$, which is what we needed to obtain.

□

For this reason one can use \mathfrak{L} to solve linear constant coefficient ODEs which can be *non-homogeneous!* Let us see how this works...

1.4. Solving ODEs. A linear, constant coefficient ODE of order n looks like:

$$\sum_{k=0}^n c_k u^{(k)}(t) = f(t).$$

In order for the solution to be unique, there must be specified initial conditions on u , that is

$$u(0), u'(0), \dots, u^{(n-1)}(0).$$

We are not requiring $f(t)$ to be the zero function, so the ODE could be *inhomogeneous*. Notoriously difficult to solve right? NOT ANYMORE! We hit both sides of the ODE with \mathfrak{L} :

$$\sum_{k=0}^n c_k \mathfrak{L}(u^{(k)})(z) = \tilde{f}(z).$$

Let's write out the left side using our proposition. First we have

$$c_0 \tilde{u}(z).$$

Then we have

$$c_1 (z\tilde{u}(z) - u(0)).$$

In general we have

$$c_k \left(z^k \tilde{u}(z) - \sum_{j=1}^k u^{(k-j)}(0) z^{j-1} \right).$$

So, if we collect all the terms with $\tilde{u}(z)$, we get

$$(c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n) \tilde{u}(z) = P(z) \tilde{u}(z),$$

$$P(z) = \sum_{k=0}^n c_k z^k.$$

Now let's collect all the rest:

$$- \sum_{k=1}^n c_k \sum_{j=1}^k u^{(k-j)}(0) z^{j-1} = Q(z).$$

This is just a polynomial also. So our ODE has been LAPLACE-TRANSFORMED into

$$P(z) \tilde{u}(z) + Q(z) = \tilde{f}(z).$$

We can solve this for $\tilde{u}(z)$:

$$\tilde{u}(z) = \frac{\tilde{f}(z) - Q(z)}{P(z)}.$$

Hence to get our solution $u(t)$ we just need to invert the Laplace transform of the right side, that is our solution will be

$$u(t) = \mathfrak{L}^{-1} \left(\frac{\tilde{f}(z) - Q(z)}{P(z)} \right).$$

1.5. Exercises for the week. These exercises will be demonstrated for you.

- (1) (Eö 55)
- (2) (7.4.6) Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $x > 0$, $0 < y < 1$ with the boundary conditions $u_x(0, y) = 0$, $u_y(x, 0) = 0$, $u(x, 1) = e^{-x}$. Express the answer as a Fourier integral.
- (3) (8.4.2) Find the temperature in a semi-infinite rod (the half-line $x > 0$) if its initial temperature is zero, and the end $x = 0$ is held at temperature 1 for $0 < t < 1$ and temperature 0 thereafter.
- (4) (Eö 14)
- (5) (Eö 45)

1.5.1. *Exercises for the week to be done oneself.*

(1) (7.4.1.c) Compute

$$\mathcal{F}_c((1+x)e^{-x}).$$

(2) (7.4.1.d) Compute

$$\mathcal{F}_s(xe^{-x}).$$

(3) (7.3.1) Use the Fourier transform to find a solution of the ordinary differential equation

$$u'' - u + 2g(x) = 0, \quad g \in \mathcal{L}^1(\mathbb{R}).$$

(4) (Eö 50)

(5) (8.4.3.a) Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c :

$$u_t = ku_{xx} \text{ for } x > 0, \quad u(x, 0) = 0, \quad u_x(0, t) = -c.$$

With the aid of the computation:

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}}e^{-a^2/(4t)}\right)(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}},$$

show that

$$u(x, t) = c\sqrt{\frac{k}{\pi}} \int_0^t s^{-1/2} e^{-x^2/(4ks)} ds.$$

(6) (8.4.1) Solve:

$$u_t = ku_{xx} - au, \quad x > 0, \quad u(x, 0) = 0, \quad u(0, t) = f(t).$$

(7) (Eö 12)

REFERENCES

- [1] Gerald B. Folland, *Fourier Analysis and Its Applications*, Pure and Applied Undergraduate Texts Volume 4, (1992).