FOURIER ANALYSIS & METHODS

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.02.27

We begin by demonstrating one additional property of the Laplace transform. Recall

$$\mathfrak{L}f(z) = \int_0^\infty e^{-zt} f(t)dt = \int_0^\infty e^{-\Re(z)t} e^{-i\Im(z)t} f(t)dt.$$

For this to be well defined we assume that f satisfies:

lap0
$$f(t) = 0 \quad \forall t < 0, \tag{1.1}$$

and that there exists $a, C > 0$ such that

lapa
$$|f(t)| \le Ce^{at} \quad \forall t \ge 0. \tag{1.2}$$

Proposition 1. *If $t^{-1}f(t)$ satisfies (lap0 1.1) and (lapa 1.2), then*

$$\mathfrak{L}(t^{-1}f(t))(z) = \int_z^\infty \mathfrak{L}f(w)dw.$$

The integral is any contour in the w-plane which starts at z along which $\Im w$ stays bounded and $\Re w \to \infty$.

Proof: Note that by (lapa 1.2), if $t^{-1}f(t)$ satisfies this, then at the point $t = 0$ apparently the function f vanishes, so that the function $t^{-1}f(t)$ is well defined. So, don't panic about this point!!! We next define the holomorphic function

$$F(z) = \int_z^\infty \widetilde{f}(w)dw.$$

Since $\widetilde{f}(w) \to 0$ when $\Re(w) \to \infty$ and $\Im(w)$ stays bounded, the fundamental theorem of calculus says that

$$F'(z) = -\widetilde{f}(z).$$

On the other hand,

$$\frac{d}{dz} \int_0^\infty t^{-1} f(t) e^{-zt} dt = \int_0^\infty -f(t)e^{-zt}dt = -\widetilde{f}(z).$$

Hence,

$$F(z) = \int_0^\infty t^{-1} f(t) e^{-zt} dt + c,$$

for some constant c. Since

$$\lim_{\Re z \to \infty} F(z) = 0 = \lim_{\Re(z) \to \infty} \int_0^\infty t^{-1} f(t) e^{-zt} dt \implies c = 0.$$

1.1. **Application of the Laplace transform to solving PDEs.** Let us consider the *telegraph equation,*

$$u_{xx} = \alpha u_{tt} + \beta u_t + \gamma u.$$

This is homogeneous, and generalizes both the heat equation ($\alpha = \gamma = 0$, and $\beta = 1$) as well as the wave equation ($\beta = \gamma = 0$, and $\alpha = 1$). According to those who know more physics than I, this corresponds to an electromagnetic signal on a cable.

We wish to solve the problem on a half line with the following boundary and initial conditions:

$$u(0,t) = f(t), \quad u(x,0) = u_t(x,0) = 0.$$

Tip 1. *If we have a half-line problem with boundary condition at $x = 0$ that is* a function of t *try using the Laplace transform in the t variable.*

We follow the tip and hit the whole PDE with the Laplace transform in the t variable. This gives

$$\widetilde{u}_{xx}(x,z) = \alpha \mathfrak{L}(u_{tt})(x,z) + \beta \mathfrak{L}(u_t)(x,z) + \gamma \widetilde{u}(x,z).$$

We use the properties of the Laplace transform and the initial conditions which say

$$u(x,0) = 0, \quad u_t(x,0) = 0,$$

so

$$\widetilde{u}_{xx}(x,z) = \alpha z^2 \widetilde{u}(x,z) + \beta z \widetilde{u}(x,z) + \gamma \widetilde{u}(x,z).$$

This is simply

$$\widetilde{u}_{xx}(x,z) = \left(\alpha z^2 + \beta z + \gamma\right) \widetilde{u}(x,z).$$

It's a second order, linear, constant coefficient, homogeneous ODE for the x variable. Let

$$q = \sqrt{\alpha z^2 + \beta z + \gamma}.$$

Our solution to the ODE is of the form

$$\widetilde{u}(x,z) = a(z) e^{qx} + b(z) e^{-qx}.$$

We have that lovely BC at $x = 0$: $u(0,t) = f(t)$. Hence,

$$\widetilde{u}(0,z) = \widetilde{f}(z) \implies a(z) + b(z) = \widetilde{f}(z).$$

Note that here we are extending f to $(-\infty, 0)$ to be identically equal to zero so that we may Laplace transform it. Assume that $\Re(q) > 0$. (If this weren't the case, just swap q and $-q$). To be able to invert the Laplace transform and get the solution to our PDE, we will not want $\widetilde{u}(x,z) \to \infty$ when $x \to \infty$. Hence, we throw out the e^{qx} solution and just use

$$\widetilde{u}(x,z) = b(z) e^{-qx}.$$

Therefore, $b(z) = \widetilde{f}(z)$. So, our Laplace-transformed solution is

$$\widetilde{u}(x,z) = \widetilde{f}(z) e^{-qx}.$$

By the properties of the Laplace transform, if we can find $g(x,t)$ such that

$$\widetilde{g}(x,z) = e^{-qx},$$

then the solution to this PDE will be

soln

$$u(x,t) = f * g(x,t) = \int_{\mathbb{R}} f(t-s)g(x,s)ds = \int_0^t f(t-s)g(x,s)ds. \tag{1.3}$$

This is because f is zero for negative times.

Now, recalling the definition of q, we are looking for

$$g(x,t) \text{ with } \widetilde{g(x,z)} = e^{-x\sqrt{\alpha z^2+\beta z+\gamma}}.$$

To find such a g, we would like to invert the Laplace transform.

1.2. **Inverting the Laplace transform.** The Laplace transform is closely related to the Fourier transform, and it is this fact, together with the FIT, that will guide our way to the LIT (Laplace Inverse Theorem).

$$\widetilde{f}(z) = \int_0^\infty f(t)e^{-zt}dt = \int_0^\infty f(t)e^{-\Re(z)t - i\Im(z)t}dt.$$

For this reason, let's define

$$g(t) = e^{-\Re(z)t}f(t),$$

so we also have

$$f(t) = e^{\Re(z)t}g(t).$$

Then

$$\mathfrak{L}f(z) = \hat{g}(\Im(z)) = \int_{\mathbb{R}} f(t)e^{-\Re(z)}e^{-i\Im(z)t}dt,$$

because $f(t) = 0$ for all $t < 0$. Let's apply the FIT to the function, g:

$$g(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)e^{i\xi t}d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \mathfrak{L}f(\Re(z)+i\xi)e^{i\xi t}d\xi.$$

To make this look less imposing, let us write $y = \xi$. So, we have

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\Re(z)+iy)e^{iyt}dy.$$

Since $f(t) = e^{\Re(z)t}g(t)$, we have

$$f(t) = e^{\Re(z)t}\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\Re(z)+iy)e^{iyt}dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\Re(z)+iy)e^{\Re(z)t+iyt}dy.$$

We would like to understand this as a complex integral. If we parametrize the vertical path for $w \in \mathbb{C}$ with $\Re(w) = \Re(z)$ by $w = \Re(z) + iy$, then $dw = idy$. We do not have an i. Hence, what we are computing is

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma_z} \widetilde{f}(w)e^{wt}dw,$$

where Γ_z is the upward vertical path along the line $\Re(w) = \Re(z)$. This is the LIT: Laplace inversion formula:

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma_z} \widetilde{f}(w)e^{wt}dw.$$

By definition of the Laplace transform, this should hold for $z \in \mathbb{C}$ with $\Re(z) > a$ where a comes from (lapa 1.2). If we naively look at this equation, we see that the left side is *independent of z*. So, the right side ought to be as well. It is.

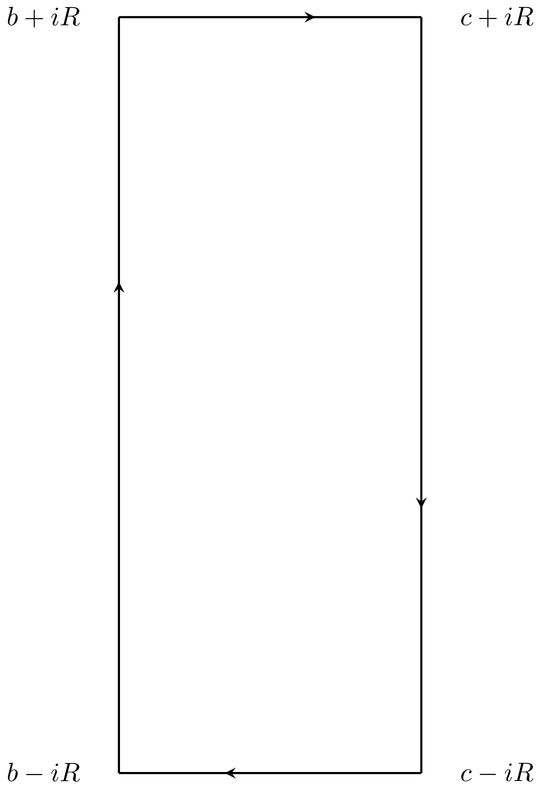

FIGURE 1. The contour over which we integral. Call the contour Γ_R. As one can see, we assume that $c > b$.

box

Theorem 2 (LIT)**.** *Assume that f is Laplace-transformable. Denote by $\widetilde{f}$ its Laplace transform. Then for $b > a$,*

$$f(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \widetilde{f}(z) e^{zt} dz.$$

Conversely, assume that $F(z)$ is analytic in $\Re(z) > a$. For $b > a$, $R > 0$, and $t \in \mathbb{R}$, let

$$f_{R,b}(t) = \frac{1}{2\pi i} \int_{b-iR}^{b+iR} F(z) e^{zt} dz.$$

Assume that for some $\alpha > 1/2$ and $C > 0$ we have

$$|F(z)| \le C(1+|z|)^{-\alpha}, \quad \forall z \in \mathbb{C} \text{ with } \Re(z) > a,$$

and assume that for some $b > a$, $f_{R,b}(t)$ converges pointwise as $R \to \infty$ to some $f(t)$ which satisfies (1.1) and (1.2). Then

$$\lim_{R\to\infty} f_{R,b}(t) = f(t) \quad \forall b > a,$$

and

$$F(z) = \mathfrak{L} f(z).$$

Proof: Let us draw and define a contour, with our amazing tikz skillz yo.

By assumption the function F is analytic inside the box, and e^{zt} is an entire function. Therefore

$$\int_{\Gamma_R} F(z)e^{zt}dz = 0.$$

So, we wish to show that the limit as $R \to \infty$ of the top and bottom integrals is zero. To obtain this, we either wave our hands like Folland or actually estimate:

$$\int_{b\pm iR}^{c\pm iR} |F(z)||e^{zt}|dz \le |c-b|e^{ct} \max_{b\le x\le c} \frac{C}{(1+|x\pm iR|)^\alpha}.$$

Above we used the fact that between $b \pm iR$ and $c \pm iR$, $|e^{zt}| \le e^{ct}$ together with the estimate assumed on F. Next, we note that

$$|x \pm iR| = \sqrt{x^2+R^2} \ge R.$$

Therefore we estimate from above by

$$|c-b|e^{ct}\frac{C}{(1+R)^\alpha} \to 0 \text{ as } R \to \infty.$$

Therefore, if for some $b > a$,

$$\lim_{R\to\infty} f_{R,b}(t) = f(t),$$

this means that

$$\lim_{R\to\infty} \int_{b-iR}^{b+iR} F(z)e^{zt}dz - \int_{c-iR}^{c+iR} F(z)e^{zt}dz = 0.$$

To see this, observe that

$$\int_{\Gamma_R} F(z)e^{zt}dz = 0 \quad \forall R.$$

Moreover, the top and bottom integrals go to zero as $R \to \infty$. Hence the sum of the left and right integrals also tends to zero as $R \to \infty$. So,

$$\lim_{R\to\infty} \int_{b-iR}^{b+iR} F(z)e^{zt}dz = \lim_{R\to\infty} \int_{c-iR}^{c+iR} F(z)e^{zt}dz \implies \lim_{R\to\infty} f_{R,b}(t) = f(t) = \lim_{R\to\infty} f_{R,c}(t).$$

Now, let us parametrize the complex integral. We use $\gamma(s) = b + is$ so $\dot\gamma(s) = ids$. Hence

$$\int_{b-iR}^{b+iR} F(z)e^{zt}dz = \int_{-R}^{R} F(b+is)e^{(b+is)t}ids = ie^{bt}\int_{-R}^{R} F(b+is)e^{ist}ds.$$

Moreover, we have assumed that

$$\lim_{R\to\infty} f_{R,b}(t) = \lim_{R\to\infty} \frac{ie^{bt}}{2\pi i}\int_{-R}^{R} F(b+is)e^{ist}ds = f(t)$$

so

$$\lim_{R\to\infty} \int_{-R}^{R} F(b+is)e^{ist}ds = 2\pi e^{-bt}f(t).$$

Let us define here

$$g_{R,b}(s) = \begin{cases} F(b+is) & |s| \le R \\ 0 & |s| > R \end{cases}.$$

Then

$$\int_{-R}^{R} F(b+is)e^{ist}ds = \int_{\mathbb{R}} g_{R,b}(s)e^{ist}ds = \widehat{g_{R,b}}(-t).$$

Moreover,

$$\lim_{R\to\infty} \widehat{g_{R,b}(-t)} = 2\pi e^{-bt} f(t).$$

Similarly

$$\lim_{R\to\infty} \widehat{g_{R,b}(t)} = 2\pi e^{bt} f(-t).$$

On the other hand,

$$\lim_{R\to\infty} g_{R,b}(s) = F(b+is).$$

By the FIT,

$$F(b+it) = \frac{1}{2\pi}\int_{\mathbb{R}} 2\pi e^{bs} f(-s) e^{its} ds.$$

It is more natural to do a change of variables, letting $\sigma = -s$, so $d\sigma = -ds$, and we get

$$F(b+it) = \int_{\sigma=\infty}^{\sigma=-\infty} e^{-b\sigma} f(\sigma) e^{-it\sigma}(-d\sigma) = \int_{-\infty}^{\infty} e^{-\sigma(b+it)} f(\sigma) d\sigma$$

$$= \int_0^{\infty} e^{-\sigma(b+it)} f(\sigma) d\sigma = \mathfrak{L} f(b+it).$$

Here we use the fact that f satisfies (1.1).

1.3. Computing an inverse Laplace transform to solve the heat equation.

For the case in which our telegraph equation is the heat equation, we have $\alpha = \gamma = 0$, and $\beta = 1$. Consequently, the square rooted polynomial in z we had named q is of the simple form:

$$q = \sqrt{z}.$$

Our Laplace-transformed solution is:

$$\widetilde{f}(z) e^{-\sqrt{z}x}.$$

Since the Laplace transform turns convolutions into multiplication, we would like to find $g(x,t)$ so that

$$\widetilde{g}(x,z) = e^{-\sqrt{z}x}.$$

Then, the solution will be given as in (1.3).

We are therefore looking for $g(x,t)$ so that

$$\widetilde{g}(x,z) = e^{-\sqrt{z}x}.$$

If we try to apply the LIT directly, we should compute

$$\int_{b-i\infty}^{b+i\infty} e^{-x\sqrt{z}} e^{zt} dz.$$

Do you know how to integrate that? I do not. It is pretty scary looking. For starters, there is the $\sqrt{z}$. This really needs to be understood using the complex logarithm which is, as the name suggests, complex.

Tip 2. *Always be careful with* $\log(z)$ *in* $\mathbb{C}$. *It is* not *entire. It is a log. Logs come from trees which have branches. Complex logs always have branches and branch cuts. You have been warned.*

So, since trying to compute the inverse Laplace transform directly seems impossible, let us try to make a reasonable guess at a function whose Laplace transform might be what we need to solve the heat equation. To solve the heat equation on $\mathbb{R}$ we used

$$e^{-x^2/(4t)}(4\pi t)^{-1/2}.$$

So, since the Laplace and Fourier transforms are closely related, and we are solving the heat equation on $[0, \infty)$, which is an unbounded interval, this is a good candidate. We shall compute its Laplace transform and see what we get. If we are super lucky, it will just give us the function we want. If we are less lucky, but still pretty lucky, the process of computing the Laplace transform together with the properties of the Laplace transform will show us how to get $g(x,t)$ whose Laplace transform is $\widetilde{g}(x,z) = e^{-\sqrt{z}x}$.

Let us therefore define:

$$\star = \int_0^\infty e^{-tz} e^{-x^2/(4t)} (4\pi t)^{-1/2} dt.$$

We are computing the Laplace transform of $\Theta(t)h(x,t)$ where

$$h(x,t) = e^{-x^2/(4t)}(4\pi t)^{-1/2}.$$

Now, we see that

$$\star = \int_0^\infty (4\pi t)^{-1/2} \exp\left(-(\sqrt{tz})^2 - \left(\frac{x}{2\sqrt{t}}\right)^2 \right) dt.$$

We do the completing the square trick in the exponent:

$$\star = \int_0^\infty (4\pi t)^{-1/2} \exp\left(-\left(\sqrt{tz} - \frac{x}{2\sqrt{t}}\right)^2 - x\sqrt{z} \right) dt$$
$$= e^{-x\sqrt{z}} \int_0^\infty \frac{1}{2\sqrt{\pi t}} \exp\left(-\left(\sqrt{tz} - \frac{x}{2\sqrt{t}}\right)^2 \right).$$

To compute this we need to use a very very clever trick. First, let us clean up our integral just a little bit to remove that pesky $\sqrt{t}$ which is getting divided. We let $s = \sqrt{t}$. Then

$$ds = \frac{dt}{2\sqrt{t}}$$

So,

$$\star = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi}} \int_0^\infty e^{-(s\sqrt{z} - x/(2s))^2} ds.$$

Theorem 3 (Cauchy & Schlömilch transform)**.**

$$\int_0^\infty af((as - b/s)^2) ds = \int_0^\infty f(y^2) dy.$$

Proof: The proof is so clever.[1]

[1] I don't know if Cauchy and Schlömilch actually had anything to do with this formula. Oscar Schlömilch was elected a foreign member of the Royal Swedish Academy of Sciences in 1862. He was a German mathematician who lived from April 13, 1823 until February 7, 1901. Augustin-Louis Cauchy was a French mathematician who lived August 21, 1789 until May 23, 1857. Did they ever meet? Why is this named after them?

We do a substitution in the integral. Let $t = \frac{b}{as}$. Then

$$dt = -\frac{b}{as^2} ds \implies -\frac{as^2}{b} dt = ds.$$

We see that

$$t^2 = \frac{b^2}{a^2 s^2} \implies \frac{a^2 s^2}{b^2} = t^{-2} \implies \frac{as^2}{b} = \frac{b}{at^2}.$$

Next, when $s \to 0$ and $s > 0$ we see that $t \to \infty$. On the other hand, when $s \to \infty$, $t \to 0$. We also see that

$$as = \frac{t}{b}, \quad -\frac{b}{s} = -ta.$$

So, let us call

$$\heartsuit = \int_0^\infty af((as - b/s)^2) ds = \int_\infty^0 af((t/b - ta)^2) \left(-\frac{b}{at^2} \right) dt$$
$$= \int_0^\infty f((t/b - at)^2) \frac{b}{t^2} dt.$$

Note that

$$(t/b - at)^2 = (-(at - t/b))^2 = (at - t/b)^2.$$

Hence we have computed

$$\heartsuit = \int_0^\infty f((at - t/b)^2) \frac{b}{t^2} dt.$$

Therefore

$$2\heartsuit = \int_0^\infty af((as - b/s)^2) ds + \int_0^\infty f((at - t/b)^2) \frac{b}{t^2} dt$$
$$= a \int_0^\infty f((as - b/s)^2) ds + b \int_0^\infty f((as - b/s)^2) \frac{ds}{s^2}.$$

As a consequence,

$$\heartsuit = \frac{1}{2} \int_0^\infty f((as - b/s)^2) \left(a + \frac{b}{s^2} \right) ds.$$

Now we let

$$y = as - \frac{b}{s} \implies dy = \left(a + \frac{b}{s^2} \right) ds.$$

We note that when $s \to 0$, $y \to -\infty$, and on the flip side, when $s \to \infty$, $y \to \infty$. Therefore

$$\heartsuit = \frac{1}{2} \int_{-\infty}^\infty f(y^2) dy = \int_0^\infty f(y^2) dy,$$

since $f(y^2)$ is an even function.

We will use the Cauchy & Schlömilch transform with

$$a = \sqrt{z}, \quad b = \frac{x}{2}, \quad f(s) = e^{-s^2}.$$

Then, it says that

$$\int_0^\infty \sqrt{z} \exp(-(as - b/s)^2) ds = \int_0^\infty \sqrt{z} \exp \left(- \left(s\sqrt{z} - \frac{x}{2s} \right)^2 \right) ds$$

$$= \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}.$$

Now we were computing

$$\star = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi}} \int_0^\infty e^{-(s\sqrt{z}-x/(2s))^2} ds = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi z}} \int_0^\infty \sqrt{z} e^{-(s\sqrt{z}-x/(2s))^2} ds$$

$$= \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

So, we have computed

$$\mathfrak{L}\left(\Theta(t)h(x,t)\right)(z) = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

This is almost what we wanted, except for the $2\sqrt{z}$ in the denominator. Here we use the properties of the Laplace transform. Consider the function:

$$\int_z^\infty \frac{e^{-x\sqrt{w}}}{2\sqrt{w}} dw = -\left.\frac{e^{-x\sqrt{w}}}{x}\right|_{w=z}^{\infty} = \frac{e^{-x\sqrt{z}}}{x}.$$

By the properties of the Laplace transform

$$\mathfrak{L}(t^{-1}f(t))(z) = \int_z^\infty \widetilde{f}(w) dw.$$

So,

$$\mathfrak{L}(t^{-1}\Theta(t)h(x,t))(z) = \int_z^\infty \frac{e^{-x\sqrt{w}}}{2\sqrt{w}} dw = \frac{e^{-x\sqrt{z}}}{x}.$$

because we computed

$$\mathfrak{L}\left(\Theta(t)h(x,t)\right)(z) = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

We can simply multiply both sides by x to get

$$\mathfrak{L}(t^{-1}x\Theta(t)h(x,t))(z) = e^{-x\sqrt{z}}$$

as desired. Let us summarize this phenomenal calculation as a theorem for future reference.

Theorem 4. *The Laplace transform of*

$$g(x,s) := \frac{x}{s}\Theta(s)h(x,s), \quad h(x,s) = \frac{1}{\sqrt{4\pi s}} e^{-\frac{x^2}{4s}}, \quad \Theta(s) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

in the variable s is

$$\mathfrak{L}(g)(x,z) = e^{-x\sqrt{z}}.$$

Therefore going back to our problem, the solution

$$u(x,t) = (f(s) * (s^{-1}x\Theta(s)h(x,s))(t) = \int_{\mathbb{R}} f(t-s)g(x,s) ds$$

$$= \int_0^t \frac{f(t-s)}{2\sqrt{\pi}s^{3/2}} x e^{-\frac{x^2}{4s}} ds.$$

This is because f is zero for negative times.

Remark 1. *One of the things I love about this class is that you begin to approach actual research mathematics. I think that must be exciting for you, because calculus (envariabelanalys) is like 300 years old. Cauchy's complex analysis is also a few hundred years old. That's not very close to actual current year 2019 math! Here is an example of how the Cauchy-Schlömilch transform is super awesome (and look, this paper is only 9 years old which is super young by research terms):*
https://arxiv.org/abs/1004.2445

1.3.1. *Hints to: exercises for the week to be done oneself.*

(1) (7.4.1.c) Compute
$$\mathcal{F}_c\left((1+x)e^{-x}\right).$$
Hint: by definition we are computing
$$\int_0^\infty (1+x)\cos(\xi x)e^{-x}dx.$$
To me, the easiest thing to do is write
$$\cos(\xi x) = \frac{e^{i\xi x}+e^{-i\xi x}}{2}.$$
Then we can combine the exponentials. So, we have two terms:
$$\int_0^\infty e^{-x}\frac{e^{i\xi x}+e^{-i\xi x}}{2}dx + \int_0^\infty xe^{-x}\frac{e^{i\xi x}+e^{-i\xi x}}{2}dx.$$
The first integral can be computed as it stands (ask if you get stuck!). For the second one use integration by parts.

(2) (7.4.1.d) Compute
$$\mathcal{F}_s\left(xe^{-x}\right).$$
Hint: Proceed in the same way as the previous problem, expanding out the sine in terms of complex exponentials.

(3) (7.3.1) Use the Fourier transform to find a solution of the ordinary differential equation
$$u'' - u + 2g(x) = 0, \quad g \in \mathcal{L}^1(\mathbb{R}).$$
Hint: Hit the whole equation with the Fourier transform in the x variable. So you are getting
$$-\xi^2\hat{u}(\xi) - \hat{u}(\xi) = -2\hat{g}(\xi).$$
Solving for $\hat{u}(\xi)$ we get
$$\hat{u}(\xi) = 2\frac{\hat{g}(\xi)}{1+\xi^2}.$$
From here, we see we got a product. The Fourier transform of a convolution results in a product. So, find a function whose Fourier transform is $\frac{1}{1+\xi^2}$. Then, you can express the solution as the convolution of $2g$ with this!

(4) (Eö 50) So we're supposed to figure out this function:
$$f(x) = \sum_{n\geq 1}\frac{\sin((2n-1)x)}{(2n-1)^3}.$$

We look to the table of Fourier series in Folland: we see that item 17 says that the Fourier series of the function $f(x) = x(\pi - |x|)$ defined this way on $(-\pi, \pi)$ and extended to be 2π periodic in $\mathbb{R}$ is

$$\frac{8}{\pi} \sum_{n \geq 1} \frac{\sin((2n-1)x)}{(2n-1)^3}.$$

So the series above is:

$\frac{\pi}{8} x(\pi - |x|)$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on $\mathbb{R}$.

(5) (8.4.3.a) Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c:

$$u_t = k u_{xx} \text{ for } x > 0, \quad u(x,0) = 0, \quad u_x(0,t) = -c.$$

With the aid of the computation:

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}} e^{-a^2/(4t)} \right)(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}},$$

show that

$$u(x,t) = c\sqrt{\frac{k}{\pi}} \int_0^t s^{-1/2} e^{-x^2/(4ks)} ds.$$

Hint: Let's hit the PDE with the Laplace transform in the t variable. We get

$$\mathfrak{L}(u_t)(x,z) = k\mathfrak{L}(u_{xx})(x,z).$$

By the properties of the Laplace transform, and the IC,

$$\mathfrak{L}(u_t)(x,z) = z\mathfrak{L}(u)(x,z) - u(x,0) = z\mathfrak{L}(u)(x,z).$$

So we have the equation:

$$\frac{z}{k} \mathfrak{L}u(x,z) = \mathfrak{L}u(x,z)_{xx}.$$

This is an ODE now for the Laplace transform of our solution. The solution is of the form:

$$\mathfrak{L}u(x,z) = A(z) e^{-x\sqrt{z/k}} + B(z) e^{x\sqrt{z/k}}.$$

We want this to be bounded for large z so we strike the second solution. The boundary condition we have is that $u_x(0,t) = -c$, so when we transform this, we want

$$\mathfrak{L}u_x(0,z) = -\mathfrak{L}(c)(z).$$

We can Laplace transform the constant function:

$$\int_0^\infty c e^{-tz} dt = \frac{c}{z}.$$

On the other hand, taking the derivative of $A(z)e^{-\sqrt{z/k}x}$ with respect to x and then setting $x = 0$ we get:

$$-\sqrt{\frac{z}{k}} A(z) \implies -\sqrt{\frac{z}{k}} A(z) = -\frac{c}{z}.$$

So, we want

$$A(z) = \frac{c\sqrt{k}}{z^{3/2}}.$$

Thus our Laplace transformed solution is:

$$\mathfrak{L}u(x,z) = \frac{c\sqrt{k}}{z^{3/2}} e^{-x\sqrt{z/k}} = c\sqrt{k}\frac{1}{z}\left(\frac{e^{-x\sqrt{z/k}}}{\sqrt{z}}\right).$$

From here on out we can follow Folland's hint and use Table 3 which says that the Laplace transform of

$$\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}(f)(z).$$

So, we have

$$\mathfrak{L}\left(\int_0^t \frac{1}{\sqrt{\pi s}} e^{-a^2/(4s)} ds\right)(z) = \frac{e^{-a\sqrt{z}}}{z\sqrt{z}}.$$

Now just deal with the constant factors and choose a correctly...

(6) (8.4.1) Solve:

$$u_t = ku_{xx} - au, \quad x > 0, \quad u(x,0) = 0, \quad u(0,t) = f(t).$$

Hint: Let's hit the PDE with the Laplace transform in the t variable and see what happens. It is a little bit different this time:

$$z\mathfrak{L}u(x,z) = k\mathfrak{L}u(x,z)_{xx} - a\mathfrak{L}u(x,z).$$

So we re-arrange and have

$$(z+a)\mathfrak{L}u(x,z) = k\mathfrak{L}u(x,z)_{xx} \implies \frac{z+a}{k}\mathfrak{L}u(x,z) = \mathfrak{L}u(x,z)_{xx}.$$

This is similar, and our solution is of the form

$$A(z)e^{-x\sqrt{(z+a)/k}} + B(z)e^{x\sqrt{(z+a)/k}}.$$

We want this to be bounded for z large, so we strike the second solution. The initial condition says we want

$$A(z) = \mathfrak{L}f(z).$$

So our Laplace-transformed solution is:

$$\mathfrak{L}f(z)e^{-x\sqrt{(z+a)/k}}.$$

This is a product. We can express our solution as a convolution if we find something whose Laplace transform is that exponential term. Let's write the exponential a little differently:

$$e^{-\frac{x}{\sqrt{k}}\sqrt{z--a}}.$$

We see that item 3 on table 3 with $c = -a$ shows that

$$\mathfrak{L}(e^{-at}f(t))(z) = \mathfrak{L}f(z--a).$$

So if we find a function whose Laplace transform is $e^{-\frac{x}{\sqrt{k}}\sqrt{z}}$ then we will be done. We see that item 27 on table 3 gives us just that:

$$\mathfrak{L}(t^{-3/2}e^{-b^2/(4t)})(z) = 2b^{-1}\sqrt{\pi}e^{-b\sqrt{z}}.$$

(We already have one thing called a running around, so I changed the name here to b). Consequently

$$\mathfrak{L}(e^{-at}t^{-3/2}e^{-b^2/(4t)})(z) = 2b^{-1}\sqrt{\pi}e^{-b\sqrt{z+a}}.$$

Now just figure out what you need b to equal to make this work. Your solution will be a convolution of f and the correct thing to make the right side equal to $e^{-\frac{x}{\sqrt{k}}\sqrt{z--a}}$.

(7) (Eö 12) We define

$$f(t) = \int_0^1 \sqrt{w}e^{w^2}\cos(wt)dw.$$

We are supposed to then somehow compute

$$\int_{\mathbb{R}} |f'(t)|^2 dt.$$

Hint: This definition of f looks remarkably like a Fourier transform of something... The right side is an $\mathcal{L}^2$ norm, so we have the Parseval (is that the right name?) formula which says that

$$\int_{\mathbb{R}} |f'(t)|^2 dt = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f'(t)}|^2 dt.$$

Then we look to Table 2 of Folland which says that

$$\widehat{f'(\xi)} = i\xi\hat{f}(\xi).$$

So we just need to compute

$$\frac{1}{2\pi}\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2 d\xi.$$

To solve this, the function f requires further inspection... it is very close to being a Fourier transform. Let us make it so. Begin by extending evenly (the presence of cosine hints at this...)

$$f(t) = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\cos(wt)dw = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}e^{-iwt}dw.$$

The reason for the last step is that the function (without the cosine) is even. So if we throw in $e^{-iwt} = \cos(-wt) + i\sin(-wt) = \cos(wt) - i\sin(wt)$ the integral with the sine will be zero since sine is odd and the rest of the integrand is zero. So we recognize

$$f(t) = \mathcal{F}\left(\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)(t).$$

By the FIT

$$\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)e^{iwt}dt = \frac{1}{2\pi}\hat{f}(-w) = \frac{1}{2\pi}\hat{f}(w).$$

This is because f is even and so it's Fourier transform is also even. So, we see that

$$\pi\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \hat{f}(w).$$

Hence, we just need to compute

$$\frac{1}{2}\int_{\mathbb{R}} w^2\left(\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)^2 dw = \frac{1}{2}\int_{-1}^1 |w|w^2e^{2w^2}dw$$

$$= \int_0^1 w^3e^{2w^2}dw.$$

Write the integrand as $(w^2)(we^{2w^2})$. Integrate by parts. It should end nicely.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).