FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Let's look at another example. Consider a circular shaped rod. Let's mathematicize it! To specify points on the rod, we just need to know the angle at the point. For this reason, we use $x \in \mathbb{R}$ for the position, where x gives us the angle at the point on the rod. We use the variable $t \geq 0$ for time. The function $u(x,t)$ is the temperature on the rod at position x at time t.

The heat equation (with no sources or sinks) tells us that:

$$u_t = k u_{xx},$$

for some constant $k > 0$. By the same little time-units-trick, we can assume that $k = 1$. So, we use the "mathematician's heat equation,"

$$u_t = u_{xx}.$$

Let's see what happens when we try Separation of Variables. We write

$$u(x,t) = f(x)g(t).$$

Plug it into the heat equation:

$$g'(t)f(x) = f''(x)g(t).$$

We want to *separate variables,* so we want all the t-dependent bits on the left say, and all the x-dependent bits on the right. This can be achieved by dividing both sides by $f(x)g(t)$,

$$\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)}.$$

We now know that both sides must be constant.

Exercise 1. *In your own words, explain why both sides of the equation must be constant.*

Now, we need to pick a side to begin... We actually have some information which is hiding inside the *geometry* of the problem. The geometry is referring to the x variable. What can you say about the angle x on the rod and the angle $x + 2\pi$ on the rod? They are the same. This means that our temperature function must be the same at x and at $x + 2\pi$. So, we must have also

$$f(x + 2\pi) = f(x).$$

We can repeat this, obtaining

$$f(x + 2\pi n) = f(x) \quad \forall n \in \mathbb{Z}.$$

This means that f is a periodic function with period equal to 2π. So, we have a bit of extra information about it. The equation for f is:

$$f''(x) = \lambda f(x)$$

for a constant λ.

Exercise 2. ***Case 1:*** *Show that if $\lambda = 0$, there is no solution to $f''(x) = 0$ which is 2π periodic, other than the constant solutions.*

Case 2: If $\lambda > 0$, then a basis of solutions is,

$$\{e^{\sqrt{\lambda}x}, e^{-\sqrt{\lambda}x}\}.$$

So, we can write

$$f(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

For the 2π periodicity to hold, we need for example

$$f(0) = f(2\pi) \implies a + b = ae^{\sqrt{\lambda}2\pi} + be^{-\sqrt{\lambda}2\pi} \implies a(e^{\sqrt{\lambda}2\pi} - 1) = b(1 - e^{-\sqrt{\lambda}2\pi})$$

$$\implies a = b\frac{(1 - e^{-\sqrt{\lambda}2\pi})}{e^{\sqrt{\lambda}2\pi} - 1}.$$

We also need

$$f(-2\pi) = f(0) \implies a + b = ae^{-\sqrt{\lambda}2\pi} + be^{\sqrt{\lambda}2\pi} \implies a(e^{-\sqrt{\lambda}2\pi} - 1) = b(1 - e^{\sqrt{\lambda}2\pi})$$

$$\implies a = b\frac{1 - e^{\sqrt{\lambda}2\pi}}{e^{-\sqrt{\lambda}2\pi} - 1}.$$

So, we got two equations for a, the should be equal:

$$a = b\frac{1 - e^{-\sqrt{\lambda}2\pi}}{e^{\sqrt{\lambda}2\pi} - 1} = b\frac{1 - e^{\sqrt{\lambda}2\pi}}{e^{-\sqrt{\lambda}2\pi} - 1}.$$

If $b = 0$ then $a = 0$ so the whole solution is the zero solution. If $b \neq 0$ then we must have

$$\frac{1 - e^{-\sqrt{\lambda}2\pi}}{e^{\sqrt{\lambda}2\pi} - 1} = \frac{1 - e^{\sqrt{\lambda}2\pi}}{e^{-\sqrt{\lambda}2\pi} - 1}.$$

Re-arranging this a bit:

$$\frac{1 - e^{-\sqrt{\lambda}2\pi}}{e^{\sqrt{\lambda}2\pi} - 1} = \frac{e^{\sqrt{\lambda}2\pi} - 1}{1 - e^{-\sqrt{\lambda}2\pi}}.$$

Call the left side $\star$. Then the right side is $\frac{1}{\star}$. So the equation is

$$\star = \frac{1}{\star} \implies \star^2 = 1 \implies \star = \pm 1.$$

Exercise 3. *Show that $\star > 0$.*

If
$$\star = 1 \implies 1 - e^{-\sqrt{\lambda}2\pi} = e^{\sqrt{\lambda}2\pi} - 1 \implies 2 = e^{\sqrt{\lambda}2\pi} + e^{-\sqrt{\lambda}2\pi}.$$
I don't like the negative exponent thing (it is really a fraction), so I am going to multiply by $e^{\sqrt{\lambda}2\pi}$. Also, doing this turns it into a quadratic equation:
$$2e^{\sqrt{\lambda}2\pi} = e^{4\pi\sqrt{\lambda}} + 1 \iff e^{4\pi\sqrt{\lambda}} - 2e^{2\pi\sqrt{\lambda}} + 1.$$
This is a quadratic equation in $\xi = e^{2\pi\sqrt{\lambda}}$. It is of the form
$$\xi^2 - 2\xi + 1 = 0,$$
and we know how to factor it
$$(\xi - 1)^2 = 0 \implies \xi = 1.$$
Since
$$\xi = e^{2\pi\sqrt{\lambda}} = 1 \iff 2\pi\sqrt{\lambda} = 0 \lightning.$$
Therefore, in the case where $\lambda > 0$, the only solution which is 2π periodic is the zero solution.

Hence, we are left with **Case 3:** $\lambda < 0$. Then, a basis of solutions is
$$\{\sin(\sqrt{|\lambda|x}), \cos(\sqrt{|\lambda|x}).$$
We need these solutions to be 2π periodic. They will be as long as $\sqrt{|\lambda|}$ is an integer. Hence, our solution
$$f_n(x) = a_n \cos(nx) + b_n \sin(nx), \quad n \in \mathbb{N}_0.$$

Exercise 4. *Show that allowing complex coefficients, it is equivalent to use a basis of solutions*
$$\{e^{\pi inx}\}_{n\in\mathbb{Z}}.$$
Find A_n and B_n so that
$$f_n(x) = A_n e^{inx} + B_n e^{-inx}.$$

Now, we can solve for the partner function, $g_n(t)$. Since
$$\frac{f_n''(x)}{f_n(x)} = -n^2 = \frac{g_n'(t)}{g_n(t)}$$
we get
$$g_n(t) = e^{-n^2 t} \text{ up to constant factor.}$$
So, we now have
$$u_n(x,t) = f_n(x)g_n(t) = e^{-n^2 t}(a_n \cos(nx) + b_n \sin(nx)).$$
It satisfies the heat equation,
$$\partial_t u_n - \partial_{xx} u_n = 0.$$
Since this equation is (1) linear and (2) homogeneous, the superposition principle allows us to smash all the solutions together into a super solution,
$$u(x,t) = \sum_{n\geq 0} u_n(x,t) = \sum_{n\geq 0} e^{-n^2 t}(a_n \cos(nx) + b_n \sin(nx)).$$
We do this because we do not know how many of the u_n functions we will need. In case we don't end up needing them all, then their coefficients will be zero, so they will just disappear on their own anyways. Let's think about the physics. The rod

has some temperature function at time $t = 0$, which we call $u_0(x)$. Then $u_0(x)$ is also a 2π periodic function. We would like

$$u(x, 0) = u_0(x) \iff \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx) = u_0(x).$$

So, given $u_0(x)$, can we find a_n and b_n so that this is true?

Fourier made the bold statement that we can do this. It took a long time to rigorously prove him right (like 100 years, because this whole theory about Hilbert spaces, measure theory, and functional analysis needed to get developed by Hilbert & his contemporaries).

1.1. **Introduction to Fourier Series of periodic functions.** If we have a finite one dimensional, connected set, then we can always mathematicize it as either (1) a bounded interval or (2) a circle. When we take a bounded interval of length 2ℓ, and we take any function whatsoever on that interval, we can always extend it to the rest of $\mathbb{R}$ to be 2ℓ periodic, by simply repeating its values from the interval. Hence, for both of these contexts we are working with periodic functions.

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p iff for all $x \in \mathbb{R}$, $f(x + p) = f(x)$.

For example, $\sin(x)$ is periodic with period 2π. Our heat equation examples, $f_n(x) = a_n \cos(nx) + b_n \sin(nx)$ are periodic with period $2\pi/n$. A small observation: I did not say the *minimal period is p.* For example, $\sin(x)$ also satisfies $\sin(x+4\pi) = \sin(x)$ for all $x \in \mathbb{R}$. So, $\sin(x)$ is also 4π periodic. In general, if a function is periodic with period p, then it's also periodic with period $2p$, $3p$, $\ldots np$ for any $n \in \mathbb{N}$ with $n \geq 1$.

Exercise 5. *Prove that any function which is p periodic (that means periodic with period p) is also np periodic for any $n \in \mathbb{N}_{\geq 1}$.*

We shall prove a super useful little lemma about periodic functions and their integrals.

Lemma 2 (Integration of periodic functions lemma)**.** *If f is periodic with period p then for any $a \in \mathbb{R}$*

$$\int_a^{a+p} f(x)dx$$

is the same.

Proof: If we think about it, we want to show that the function

$$g(a) := \int_a^{a+p} f(x)dx$$

is a constant function. This looks awfully similar to the fundamental theorem of calculus. In any decent proof, we need to use the hypotheses of the lemma. So, we're going to need to use the assumption that f is periodic with period p, which tells us that

$$f(a + p) - f(a) = 0.$$

Now, since we want to consider a as a variable, we don't want it at both the top and the bottom of the integral defining g. Instead, we can use linearity of integration to write

$$g(a) = \int_0^{a+p} f(x)dx - \int_0^a f(x)dx.$$

Then, using the fundamental theorem on each of the two terms on the right,

$$g'(a) = f(a+p) - f(a) = 0.$$

Above, we use the fact that f is periodic with period p. Hence, $g'(a) \equiv 0$ for all $a \in \mathbb{R}$. This tells us that g is a constant function, so its value is the same for all $a \in \mathbb{R}$.

So you survived a bit of theory, now let's return to our physical motivation! We wanted to find coefficients so that the $u(x,t)$ we found to solve the heat equation would match up with the initial data, $u_0(x)$. If it does, then (using some advanced PDE theory beyond the scope of this humble course), $u(x,t)$ is indeed THE UNIQUE solution to the heat equation with initial data $u_0(x)$. Hence, $u(x,t)$ actually tells us the temperature on the rod at position x at time t. Cool. So, setting $t=0$ in the definition of $u(x,t)$ we want

vx

$$u_0(x) = \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx). \quad (1.1)$$

It is totally equivalent to work with complex exponentials, because

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}.$$

Exercise 6. *Show that we can write $u_0(x)$ as a series above in (1.1) if and only if we can write*

$$u_0(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Moreover, show that

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1, \quad c_n = \frac{1}{2}(a_n + ib_n), n \leq -1.$$

Finally, use this to show that

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n},\ n \geq 0, \quad b_n = i(c_n - c_{-n}),\ n \geq 0.$$

It is slightly more convenient for these purposes to do the calculation using the $\{e^{inx}\}_{n \in \mathbb{Z}}$ basis. This will be elucidated in a moment. The equation we want to obtain is:

$$u_0(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

The object on the right is a sum of coefficients $c_n \in \mathbb{C}$ times functions e^{inx}. It is simply a linear combination of the functions e^{inx}. If we could show that in a suitable sense these functions for a sort of "basis" then we should be able to expand our function u_0 in terms of this basis. Sure, the basis is infinite, so, you've graduated to "linear algebra for adults," in which your vectors are now infinite dimensional. [1] To continue with the linear algebra concept, we need a notion of orthogonality.

[1] Grigori Rozenblioum, who taught this class for many years, and is in general an awesome mathematician, used to say "If you can pass this course, then you've earned the right to buy Vodka at Systembolaget, regardless of your actual age."

This is obtained using something called a scalar product, or dot product, or inner product: they all mean the same thing.

Since we are doing this for functions, we want to be able to say when functions are orthogonal to each other, to define norms of functions, and to project arbitrary functions onto "basis functions." The scalar product which will allow us to do this is:

Definition 3. For two functions, f and g, which are real or complex valued functions defined on $[a,b] \subset \mathbb{R}$, we define their scalar product to be

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

We say that f and g are orthogonal if $\langle f, g \rangle = 0$. We define the $L^2([a,b])$ *norm* of a function to be

$$||f||_{L^2([a,b])} = \sqrt{\langle f, f \rangle}.$$

OBS! Learn this definition right now!!!! It is really important. Every detail:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx, \quad ||f||^2 = \langle f, f \rangle.$$

Now, if you wonder *why* it is defined this way, that is because defining things this way has the very pleasant consequence that it *works.* Meaning, when we define things this way, we are able to use the separation of variables technique to solve the PDEs.

2. Exercises for this week

These exercises are found in [folland 1].

2.1. Exercises to be demonstrated.

(1) Derive pairs of ordinary differential equations from the following partial differential equations by separation of variables, or show that it is not possible:

$$yu_{xx} + u_y = 0$$
$$x^2u_{xx} + xu_x + u_{yy} + u = 0$$
$$u_{xx} + u_{xy} + u_{yy} = 0$$
$$u_{xx} + u_{xy} + u_y = 0$$

(2) Derive sets of three ordinary differential equations from the following partial differential equations by separation of variables:

$$yu_{xx} + xu_{yy} + yxu_{zz} = 0$$
$$x^2u_{xx} + xu_x + u_{yy} + x^2u_{zz} = 0$$

(3) Solve:

$$u_t = \frac{1}{10}u_{xx}, \quad u_x(0,t) = u_x(\pi,t) = 0$$
$$u(x,0) = 3 - 4\cos(2x), \quad 0 < x < \pi.$$

OBS! If you are trying to do this yourself, it's going to be hard to get the initial condition as of now, because we have not finished learning how to create Fourier series. So, as of Wednesday, it is sufficient if you understand how to solve the equation using the techniques we have so far (separation of variables + superposition). How to obtain the coefficients will be explained in the exercises, and we will also do this on Friday.

(4) By separation of variables, derive the family of solutions

$$u^{\pm}_{mn}(x,y,z) = \sin(m\pi x)\cos(n\pi y)\exp\left(\pm\sqrt{m^2+n^2}\pi z\right)$$

to the equation

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad u(0,y,z) = u(1,y,z) = u_y(x,0,z) = u_y(x,1,z) = 0.$$

2.2. **Exercises to do yourself.**

(1) The object of this exercise is to derive *d'Alembert's formula* for the general solution of the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$.
   (a) Show that if $u(y,z) = f(y) + g(z)$ where f and g are C^2 functions of one variable, the u satisfies

   $$u_{yz} = 0.$$

   Conversely, show that every C^2 solution of this equation is of this form. (Hint: If $v_y = 0$ then v is independent of y).
   (b) Let $y = x - ct$ and $z = x + ct$. Use the chain rule to show that

   $$u_{tt} - c^2 u_{xx} = -4c^2 u_{yz}.$$

   (c) Explain why the general C^2 solution of the wave equation is

   $$u(x,t) = \phi(x - ct) + \psi(x + ct)$$

   where ϕ and ψ (the Greek functions) are C^2 functions of one variable. Explain why physically $\phi(x - ct)$ represents a wave traveling to the right with speed c and $\psi(x+ct)$ represents a wave traveling to the left with speed c.
   (d) Show that the solution of the initial value problem for the wave equation,

   $$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

   is

   $$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy.$$

(2) Solve:

$$u_{tt} = 9u_{xx}, \quad u(0,t) = u(1,t) = 0,$$
$$u(x,0) = 2\sin(\pi x) - 3\sin(4\pi x), \quad u_t(x,0) = 0, \quad (0 < x < 1).$$

OBS! It is going to be hard to get the initial condition as of now, because we have not finished learning how to create Fourier series. So, as of Wednesday, it is sufficient if you understand how to solve the equation using the techniques we have so far (separation of variables + superposition). How to obtain the coefficients will be explained in the exercises, and we will also do this on Friday.

(3) By separation of variables, derive the solutions

$$u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$$

to:

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

(4) Use separation of variables to find an infinite family of independent solutions of
$$u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(l,t) = 0.$$
This represents heat flow in a rod with one end held at temperature zero and the other end insulated.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).