

# FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications*, by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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We have seen how the process of solving PDEs like the heat and wave equation often leads to a set of functions which comprise an orthogonal basis for  $\mathcal{L}^2$  or a weighted  $\mathcal{L}^2$  space. These basis functions generally come from separation of variables. When we solve the “space” part of the PDE, we very often end up solving a type of SLP. The easiest examples are:

$$\begin{aligned} f'' &= \lambda f, & f(a) = 0 = f(b), & \text{ for } f \text{ defined on the interval, } [a, b] \\ f'' &= \lambda f, & f'(a) = 0 = f'(b), & \text{ for } f \text{ defined on the interval, } [a, b] \\ f'' &= \lambda f, & f(a) = 0 = f'(b), & \text{ for } f \text{ defined on the interval, } [a, b] \\ f'' &= \lambda f, & f'(a) = 0 = f(b), & \text{ for } f \text{ defined on the interval, } [a, b]. \end{aligned}$$

A more challenging example comes from solving the heat and wave equations on a circular sector. There, when we did separation of variables, we got the nice type of SLP above for the angular variable ( $\theta$ ), and we got a more complicated SLP for the radial variable. This turned into a Bessel equation. We used the initial data to determine the coefficients in our series expansion, by writing the initial data as a Fourier-Bessel type series.

**1.1. Solving PDEs and special functions: introducing the French polynomials.** In *other* geometric settings, this same process will lead to *other* special functions. In the last part of this course, based on chapter 6 in Folland, we will look at *the French polynomials*,

- (1) Legendre polynomials
- (2) Hermite polynomials
- (3) Laguerre polynomials

1.1.1. *Legendre polynomials.* These French polynomials arise from using spherical coordinates to solve the wave and heat equations on a three-dimensional sphere.

1.1.2. *Hermite polynomials.* These French polynomials arise from using parabolic coordinates to solve the wave and heat equations in a parabolic shaped region.

1.1.3. *Laguerre polynomials.* These French polynomials arise from the quantum mechanics of the hydrogen atom.

**1.2. Orthogonal polynomials general theory.** For the purpose of this course, it is most important that you learn how to *use* the orthogonal polynomials. Depending on how much time we have, we may go into the details of the origins of the French polynomials, but these details are rather complicated, and we will not be examined. So, we prioritize that which shall be examined. The following proposition will be useful.

**Proposition 1.** *Assume that  $\{p_n\}_{n \in \mathbb{N}}$  is a sequence of polynomials such that  $p_n$  is of degree  $n$  for each  $n$ . Assume that  $p_0 \neq 0$ . Then for each  $k \in \mathbb{N}$ , any polynomial of degree  $k$  is a linear combination of  $\{p_j\}_{j=0}^k$ .*

**Proof:** The proof is by induction of course! If  $q_0$  is a polynomial of degree 0, then we may simply write

$$q_0 = \frac{q_0}{p_0} p_0.$$

This is okay because  $p_0$  is degree zero, so it is a constant, and  $p_0 \neq 0$ , so the coefficient  $q_0/p_0$  is also a constant. Assume that we have verified the proposition for all  $0, 1, \dots, k$ . We wish to show that it holds for  $k+1$ . So, let  $q$  be a polynomial of degree  $k+1$ . This means that

$$q(x) = ax^{k+1} + l.o.t. \quad l.o.t. \text{ means lower order terms}$$

has

$$a \neq 0.$$

Moreover, since  $p_{k+1}$  is of degree  $k+1$  (not of a lower degree), it is of the form

$$p_{k+1} = bx^{k+1} + l.o.t., \quad b \neq 0.$$

So, let us consider

$$q(x) - \frac{a}{b} p_{k+1}(x) = p(x) \text{ which is degree } k.$$

By induction,  $p$  is a linear combination of  $p_0, \dots, p_k$ . Therefore

$$q(x) = \frac{a}{b} p_{k+1} + \sum_{j=0}^k c_j p_j,$$

for some constants  $\{c_j\}_{j=0}^k$ .



**Proposition 2.** *Let  $\{p_k\}_{k=0}^{\infty}$  be a set of polynomials such that each  $p_k$  is of degree  $k$ , and  $p_0 \neq 0$ . Moreover, assume that they are  $\mathcal{L}^2$  orthogonal on a finite bounded interval  $[a, b]$ . Then these polynomials comprise an orthogonal basis of  $\mathcal{L}^2$  on the interval  $[a, b]$ .*

**Proof:** Assume that some  $f \in \mathcal{L}^2$  on the interval is orthogonal to all of these polynomials. Therefore by the preceding proposition,  $f$  is orthogonal to *all* polynomials. To see this, note that if  $p$  is a polynomial of degree  $n$ , then there exist numbers  $c_0, \dots, c_n$  such that

$$p = \sum_{j=0}^n c_j p_j \implies \langle f, p \rangle = \sum_{j=0}^n \overline{c_j} \langle f, p_j \rangle = 0.$$

We shall use the fact that continuous functions are dense in  $\mathcal{L}^2$ . Therefore given  $\varepsilon > 0$ , there exists a continuous function,  $g$ , such that

$$\|f - g\| < \frac{\varepsilon}{2(\|f\| + 1)}.$$

Next, we use the Stone-Weierstrass Theorem which says that all continuous functions on bounded intervals can be approximated by polynomials. Therefore, there exists a polynomial  $p$  such that

$$\|g - p\| < \frac{\varepsilon}{2(\|f\| + 1)}.$$

Finally, we compute

$$\begin{aligned} \|f\|^2 &= \langle f, f \rangle = \langle f - g + g - p + p, f \rangle = \langle f - g, f \rangle + \langle g - p, f \rangle + \langle p, f \rangle \\ &= \langle f - g, f \rangle + \langle g - p, f \rangle. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\|f\|^2 \leq \|f - g\|\|f\| + \|g - p\|\|f\| < \frac{\|f\|\varepsilon}{2(\|f\| + 1)} + \frac{\|f\|\varepsilon}{2(\|f\| + 1)} < \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, this shows that  $\|f\| = 0$ . Hence by the three equivalent conditions to be an orthogonal basis, we have that the polynomials are an orthogonal basis of  $\mathcal{L}^2$  on the interval.



**1.3. Best approximations.** We recall a slight variation of the best approximation theorem:

**Theorem 3.** *Let  $\{\phi_n\}_{n \in \mathbb{N}}$  be an orthonormal set set in a Hilbert space,  $H$ . If  $f \in H$ ,*

$$\|f - \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n\| \leq \|f - \sum_{n \in \mathbb{N}} c_n \phi_n\|, \quad \forall \{c_n\}_{n \in \mathbb{N}} \in \ell^2,$$

*and = holds  $\iff c_n = \langle f, \phi_n \rangle$  holds  $\forall n \in \mathbb{N}$ . More generally, let  $\{\phi_n\}_{n=0}^N$  be an orthogonal, non-zero set in a Hilbert space  $H$ . Then,*

$$\|f - \sum_{n=0}^N \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \phi_n\| \leq \|f - \sum_{n=0}^N c_n \phi_n\|, \quad \forall \{c_n\}_{n=0}^N \in \mathbb{C}^{N+1}.$$

*Equality holds if and only if*

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}, \quad n = 0, \dots, N.$$

How to prove it? The only difference is the last part, but we can use the proof of the first part. Define  $\psi_n = 0$  for  $n > N$ . Next define

$$\psi_n = \frac{\phi_n}{\|\phi_n\|}, \quad n = 0, \dots, N.$$

Repeat the argument in the proof of the best approximation theorem using  $\{\psi_n\}_{n \in \mathbb{N}}$  instead of  $\phi_n$ .

$$\|f - \sum_{n \in \mathbb{N}} c_n \psi_n\|^2 = \|f - \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n + \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n - \sum_{n \in \mathbb{N}} c_n \psi_n\|^2$$

$$= \|f - \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n\|^2 + \left\| \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n - \sum_{n \in \mathbb{N}} c_n \psi_n \right\|^2 + 2\Re \langle f - \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n, \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n - \sum_{n \in \mathbb{N}} c_n \psi_n \rangle.$$

The scalar product

$$\langle f - \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n, \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n - \sum_{n \in \mathbb{N}} c_n \psi_n \rangle = \langle f, \sum_{n \in \mathbb{N}} (\hat{f}_n - c_n) \Psi_n \rangle - \sum_{n \in \mathbb{N}} \hat{f}_n \langle \psi_n, \sum_{m \in \mathbb{N}} (\hat{f}_m - c_m) \Psi_n \rangle.$$

By the orthogonality and definition of  $\Psi_n$ , and the definition of  $\hat{f}_n$ ,

$$\begin{aligned} &= \sum_{n \in \mathbb{N}} \hat{f}_n \overline{(\hat{f}_n - c_n)} - \sum_{n \in \mathbb{N}} \hat{f}_n \sum_{m \in \mathbb{N}} \overline{(\hat{f}_m - c_m)} \langle \psi_n, \psi_m \rangle \\ &= \sum_{n \in \mathbb{N}} \hat{f}_n \overline{(\hat{f}_n - c_n)} - \sum_{n \in \mathbb{N}} \hat{f}_n \overline{(\hat{f}_n - c_n)} = 0. \end{aligned}$$

Therefore

$$\begin{aligned} \|f - \sum_{n \in \mathbb{N}} c_n \psi_n\|^2 &= \|f - \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n\|^2 + \left\| \sum_{n \in \mathbb{N}} \hat{f}_n \psi_n - \sum_{n \in \mathbb{N}} c_n \psi_n \right\|^2 \\ &= \|f - \sum_{n=0}^N \hat{f}_n \psi_n\|^2 + \sum_{n=0}^N |\hat{f}_n - c_n|^2 \leq \|f - \sum_{n=0}^N \hat{f}_n \psi_n\|^2, \end{aligned}$$

with equality if and only if  $c_n = \hat{f}_n$  for all  $n$ . Since

$$\sum_{n=0}^N \hat{f}_n \psi_n = \sum_{n=0}^N \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \phi_n,$$

this completes the proof. ☞

1.3.1. *Applications: best approximation problems.* This shows us that if we have a finite orthogonal set of non-zero vectors in a Hilbert space, then for any element of that Hilbert space, the best approximation of  $f$  in terms of those vectors is given by

$$\sum_{n=0}^N \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \phi_n.$$

Here is the setup of questions which can be solved using this theory. Either:

- (1) You are given functions defined on an interval which are  $\mathcal{L}^2$  orthogonal on that interval (possibly with respect to a weight function which is also specified). Either you recognize that they are orthogonal because you've seen them before (like sines, cosines, from problems you have solved previously) or you *compute* that they are  $\mathcal{L}^2$  orthogonal on the interval. Then, you are asked to find the numbers  $c_0, c_1, \dots, c_N$  so that the  $\mathcal{L}^2$  norm, or the weighted  $\mathcal{L}^2$  norm of  $f - \sum_{k=0}^N c_k \phi_k$  is minimized, where the function  $f$  is also specified.
- (2) You are asked to find the polynomial of at most degree  $N$  such that the  $\mathcal{L}^2$  norm (or weighted  $\mathcal{L}^2$  norm) of  $f - p$  where  $p$  is a polynomial is minimized.

In the first case, you compute

$$c_k = \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2}.$$

In the second case you need to *build up a set of orthogonal or orthonormal polynomials*. Then, you let  $\phi_k$  be defined to be the polynomial of degree  $k$  you have built. Proceed the same as in the first case, and your answer shall be

$$\sum_{k=0}^N c_k \phi_k.$$

If you don't like the thought of building up a set of orthogonal polynomials, if you are lucky, then it may be possible to suitably modify some of the French polynomials to be orthogonal on the interval under investigation, with respect to the (possibly weighted)  $\mathcal{L}^2$  norm. So, we shall proceed to study the French polynomials. Depending on how much time we have, we may also be able to get into their "origin stories."

**1.4. The Legendre polynomials.** The *Legendre polynomials*, are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} ((x^2 - 1)^n).$$

OMG like why on earth are they defined in such a bizarre way, right? What did you expect, they are French polynomials! Of course they are not defined in some simple way, mais non, they must be all fancy and shrouded in mystery and intrigue. Actually though, the reason comes from the PDE in which they arise as solving one part of the separation of variables for the heat and wave equations in three dimensions using spherical coordinates. First, let us do a small calculation involving these polynomials:

$$(x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (x^2)^k = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}.$$

Therefore, if we differentiate  $n$  times, only the terms with  $k \geq n/2$  survive. Differentiating a term  $x^{2k}$  once we get  $2kx^{2k-1}$ . Differentiating  $n$  times gives

$$\frac{d^n}{dx^n} (x^{2k}) = x^{2k-n} \prod_{j=0}^{n-1} (2k - j).$$

If we want to be really persnickety, we prove this by induction. For  $n = 1$ , we get that

$$(x^{2k})' = 2kx^{2k-1}.$$

Which is correct. If we assume the formula is true for  $n$ , then differentiating  $n + 1$  times using the formula for  $n$  we get

$$(2k - n)x^{2k-(n+1)} \prod_{j=0}^{n-1} (2k - j) = x^{2k-(n+1)} \prod_{j=0}^n (2k - j).$$

See, it is correct. As a result,

$$P_n(x) = \frac{1}{2^n n!} \sum_{k \geq n/2}^n (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k - j).$$

So, we see that this is indeed a polynomial of degree  $n$ .

## 1.5. Hints for the exercises to be done oneself.

- (1) (5.5.5) Solve the problem

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r, \theta, z) : 0 \leq r \leq b, 0 \leq z \leq l\}$$

$$u(r, \theta, 0) = 0, \quad u(r, \theta, l) = g(r, \theta), \quad u(b, \theta, z) = 0.$$

Hint: There is only one inhomogeneous part of the equation, and that is the boundary condition when  $z = l$ . Otherwise, observe that since we are in a cylinder, the function must be  $2\pi$  periodic in the theta variable. So, let us separate variables writing  $u = R(r)\Theta(\theta)Z(z)$ . Put this into the PDE. First solve for the theta dependent function. I am guessing you will get either  $e^{in\theta}$  for  $n \in \mathbb{Z}$  or  $\sin(n\theta)$  and  $\cos(n\theta)$ , and these are equivalent to each other... Next, I would solve for the  $R$  function. This has the zero boundary condition:  $R(b) = 0$ . So, I am guessing you will get  $J_n(z_{n,k}r/b)$  where  $z_{n,k}$  is the  $k^{\text{th}}$  positive zero of the Bessel function  $J_n$  for  $n \in \mathbb{N}$ . Last but not least, use these to solve for your  $Z$  function. Since the PDE is homogeneous, smash them all together into a super-solution using superposition. Use the condition  $u(r, \theta, l) = g(r, \theta)$  to specify what the constants in your solution need to be.

- (2) (5.2.4) Demonstrate the identity:

$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$

Hint: Use the recurrence formulas. Integrating by parts is a reasonable idea as well.

- (3) (5.2.8) Prove the reduction formula:

$$\int_0^x s^n J_0(s)ds = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int_0^x s^{n-2}J_0(s)ds.$$

Hint: Integrate by parts, using the facts that  $(xJ_1)' = xJ_0$  and  $J_0' = -J_1$ .

- (4) (5.4.2) Expand the function
- $f(x) = b^2 - x^2$
- in a Fourier-Bessel series of the form

$$\sum_{k \geq 1} c_k J_0(\lambda_k x/b),$$

where  $\lambda_k$  is the  $k^{\text{th}}$  positive zero of the Bessel function  $J_0$ . OBS! Remember to integrate with respect to  $x dx$ , polar coordinate style.

Hint: the coefficients will be

$$\frac{\int_0^b (b^2 - x^2)J_0(\lambda_k x/b)x dx}{\int_0^b |J_0(\lambda_k x/b)|^2 x dx}.$$

According to Folland, the denominator is going to be

$$\frac{b^2}{2} J_1(\lambda_k)^2.$$

You may use this to obtain some clever simplification.

- (5) (6.4.6) Let
- $f(x) = 1$
- for
- $x > 0$
- and 0 for
- $x < 0$
- . (Heavyside function basically). Expand
- $f$
- in a series of Hermite polynomials.

Hint: the coefficients will be

$$\frac{1}{\|H_n\|^2} \int_0^\infty H_n(x)e^{-x^2} dx.$$

The part  $\|H_n\|^2 = 2^n n! \sqrt{\pi}$ . For the other part, use the fact that:  $e^{-x^2} H_n = -(e^{-x^2} H_{n-1})'$ . Next, use the generating function for the Hermite polynomials to evaluate  $H_n(0)$ .

- (6) Eö 39. Hint: This is a best approximation problem in disguise! What you are trying to minimize is:

$$\int_0^1 |x^3 - (-ax^2 - bx - c)|^2 dx.$$

So, you are looking for the polynomial of degree two which best approximates the function  $f(x) = x^3$ .

- (7) Eö 41. Hint: use the generating function for the Hermite polynomials. Differentiate both sides with respect to  $x$  and then set  $x = 0$ . Expand the right side in a Taylor series and then equate the coefficients.

#### REFERENCES

- [1] Gerald B. Folland, *Fourier Analysis and Its Applications*, Pure and Applied Undergraduate Texts Volume 4, (1992).