FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Theorem 1. *The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$, and*

$$||P_n||^2 = \frac{2}{2n+1}.$$

Proof: We first prove the orthogonality. Assume that $n > m$. Then, since they have this constant stuff out front, we compute

$$2^n n! 2^m m! \langle P_n, P_m \rangle = \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m dx.$$

Let us integrate by parts once:

$$= \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m \Big|_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

Consider the boundary term:

$$\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}}(x-1)^n(x+1)^n.$$

This vanishes at $x = \pm 1$, because the polynomial vanishes to order n whereas we only differentiate $n-1$ times. So, we have shown that

$$2^n n! 2^m m! \langle P_n, P_m \rangle = -\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

We repeat this $n-1$ more times. We note that for all $j < n$,

$$\frac{d^j}{dx^j}(x^2-1)^n \text{ vanishes at } x = \pm 1.$$

For this reason, all of the boundary terms from integrating by parts vanish. So, we just get

$$(-1)^n \int_{-1}^{1} (x^2-1) \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^m dx = (-1)^n \int_{-1}^{1} (x^2-1) \frac{d^n}{dx^n} \frac{d^m}{dx^m}(x^2-1)^m dx$$

Remember that $n > m$. We computed that $\frac{d^m}{dx^m}(x^2-1)^m$ is a polynomial of degree m. So, if we differentiate it more than m times we get zero. So, we're integrating zero! Hence it is zero.

For the second part, we use the formula we computed for

$$P_n(x) = \frac{1}{2^n n!} \sum_{k \geq n/2}^{n} (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

Differentiating n times gives us just the term with the highest power of x, so we have

$$\frac{d^n}{dx^n} P_n(x) = \frac{1}{2^n n!} n! \prod_{j=0}^{n-1} (2n-j) = \frac{(2n)!}{2^n n!}.$$

Consequently,

$$\langle P_n, P_n \rangle = (-1)^n \frac{1}{2^n n!} \frac{(2n)!}{2^n n!} \int_{-1}^{1} (x^2-1)^n dx = (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx$$

$$= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^{2k} dx$$

$$= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^{n-k} \frac{x^{2k+1}}{2k+1} \binom{n}{k} \Big|_0^1$$

$$= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{2k+1}$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{1}{2k+1}.$$

This looks super complicated. Apparently by some miracle of life

$$\int_0^1 (1-x^2)^n dx = \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+3/2)}.$$

Since

$$\langle P_n, P_n \rangle = (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx = \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (1-x^2)^n dx,$$

we get

$$\frac{\Gamma(n+1)\Gamma(1/2) 2(2n)!}{2^{2n}(n!)^2 \Gamma(n+3/2)}.$$

We use the properties of the Γ function together with the fact that $\Gamma(1/2) = \sqrt{\pi}$ to obtain

$$\frac{\sqrt{\pi} 2(2n)!}{2^{2n} n! (n+1/2)\Gamma(n+1/2)}.$$

Let us consider

$$2(n+1/2)\Gamma(n+1/2) = (2n+1)\Gamma(n+1/2).$$

Next consider

$$2(n-1/2)\Gamma(n-1/2) = (2n-1)\Gamma(n-1/2).$$

Proceeding this way, the denominator becomes

$$2^n n! (2n+1)(2n-1) \dots 1 \sqrt{\pi}.$$

However, now looking at the first part

$$2^n n! = 2n(2n-2)(2n-4) \dots 2.$$

So together we get

$$(2n+1)!\sqrt{\pi}.$$

Hence putting this in the denominator of the expression we had above, we have

$$\frac{\sqrt{\pi}2(2n)!}{(2n+1)!\sqrt{\pi}} = \frac{2}{2n+1}.$$

Corollary 2. *The Legendre polynomials are an orthogonal basis for $\mathcal{L}^2$ on the interval* $[-1,1]$.

Theorem 3. *The even degree Legendre polynomials $\{P_{2n}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$. The odd degree Legendre polynomials $\{P_{2n+1}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$.*

Proof: Let f be defined on $[0,1]$. We can extend f to $[-1,1]$ either evenly or oddly. First, assume we have extended f evenly. Then, since $f \in \mathcal{L}^2$ on $[0,1]$,

$$\int_{-1}^{1} |f_e(x)|^2 dx = 2\int_0^1 |f(x)|^2 dx < \infty.$$

Therefore f_e is in $\mathcal{L}^2$ on the interval $[-1,1]$. We have proven that the Legendre polynomials are an orthogonal basis. So, we can expand f_e in a Legendre polynomial series, as

$$\sum_{n\geq 0} \hat{f}_e(n) P_n,$$

where

$$\hat{f}_e(n) = \frac{\langle f_e, P_n\rangle}{||P_n||^2.}$$

By definition,

$$\langle f_e, P_n\rangle = \int_{-1}^{1} f_e(x)P_n(x)dx.$$

Since f_e is even, the product $f_e(x)P_n(x)$ is an *odd* function whenever n is odd. Hence all of the odd coefficients vanish. Moreover,

$$\langle f_e, P_{2n}\rangle = 2\int_0^1 f(x)P_{2n}(x))dx.$$

We also have

$$||P_{2n}||^2 = 2\int_0^1 |P_{2n}(x)|^2 dx.$$

Consequently

$$f = \sum_{n\in\mathbb{N}} \left(\frac{\int_0^1 f(x)P_{2n}(x)dx}{\int_0^1 |P_{2n}(x)|^2 dx} \right) P_{2n}.$$

We can also extend f oddly. This odd extension satisfies

$$\int_{-1}^{1} |f_o(x)|^2 dx = \int_{-1}^{0} |f_o(x)|^2 dx + \int_0^1 |f_o(x)|^2 dx = 2\int_0^1 |f_o(x)|^2 dx < \infty.$$

So, the odd extension is also in $\mathcal{L}^2$ on the interval $[-1, 1]$. We can expand f_o in a Legendre polynomial series, as

$$\sum_{n \geq 0} \hat{f}_o(n) P_n,$$

where

$$\hat{f}_o(n) = \frac{\langle f_o, P_n \rangle}{||P_n||^2.}$$

By definition,

$$\langle f_o, P_n \rangle = \int_{-1}^{1} f_o(x) P_n(x) dx.$$

Since f_o is odd, the product $f_o(x)P_n(x)$ is an *odd* function whenever n is *even.* Hence all of the even coefficients vanish. Moreover,

$$\langle f_o, P_{2n+1} \rangle = 2 \int_0^1 f(x) P_{2n+1}(x)) dx,$$

because the product of two odd functions is an even function. We also have

$$||P_{2n+1}||^2 = \int_{-1}^{0} |P_{2n+1}(x)|^2 dx + \int_0^1 |P_{2n+1}(x)|^2 dx = 2 \int_0^1 |P_{2n+1}(x)|^2 dx.$$

Consequently

$$f = \sum_{n \in \mathbb{N}} \left(\frac{\int_0^1 f(x) P_{2n+1}(x) dx}{\int_0^1 |P_{2n+1}(x)|^2 dx} \right) P_{2n+1}.$$

1.1. **Hermite polynomials.**

Definition 4. *The Hermite polynomials are defined to be*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Proposition 5. *The Hermite polynomials are polynomials with the degree of H_n equal to n.*

Proof: The proof is by induction. For $n = 0$, this is certainly true, as $H_0 = 1$. Next, let us assume that

$$\frac{d^n}{dx^n} e^{-x^2} = p_n(x) e^{-x^2},$$

is true for a polynomial, p_n which is of degree n. Then,

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = \frac{d}{dx} \left(p_n(x) e^{-x^2} \right) = p_n'(x) e^{-x^2} - 2x p_n(x) e^{-x^2} = \left(p_n'(x) - 2x p_n(x) \right) e^{-x^2}.$$

Let

$$p_{n+1} = p_n'(x) - 2x p_n(x).$$

Then we see that since p_n is of degree n, p_{n+1} is of degree $n + 1$. Moreover

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = p_{n+1}(x) e^{-x^2}.$$

So, in fact, the Hermite polynomials satisfy:

$$H_0 = 1, \quad H_{n+1} = - \left(H_n'(x) - 2x H_n(x) \right).$$

Proposition 6. *The Hermite polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function e^{-x^2}. Moreover, with respect to this weight function $||H_n||^2 = 2^n n!\sqrt{\pi}$.*

Proof: Assume $n > m \geq 0$. We compute

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}dx = \int_{\mathbb{R}} (-1)^n \frac{d^n}{dx^n}e^{-x^2}H_m(x)dx.$$

We use integration by parts n times, noting that the rapid decay of e^{-x^2} kills all boundary terms. We therefore get

$$\int_{\mathbb{R}} e^{-x^2}\frac{d^n}{dx^n}H_m(x)dx = 0.$$

This is because the polyhomial, H_m, is of degree $m < n$. Therefore differentiating it n times results in zero. Finally, for $n = m$, we have by the same integration by parts,

$$\int_{\mathbb{R}} H_n^2(x)e^{-x^2}dx = \int_{\mathbb{R}} e^{-x^2}\frac{d^n}{dx^n}H_n(x)dx.$$

The n^{th} derivative of H_n is just the n^{th} derivative of the highest order term. By our preceding calculation, the highest order term in H_n is

$$(2x)^n.$$

Differentiating n times gives

$$2^n n!.$$

Thus

$$\int_{\mathbb{R}} H_n^2(x)e^{-x^2}dx = 2^n n! \int_{\mathbb{R}} e^{-x^2}dx = 2^n n!\sqrt{\pi}.$$

We may wish to use the following lovely fact, but we shall not prove it.

Theorem 7. *The Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-x^2}.*

What we shall prove, however, is a theory item concerning the Hermite polynomials.

1.1.1. *The generating function for the Hermite polynomials.* This is similar to the analogous result for the Bessel functions, but with a bit of a twist.

Theorem 8. *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2},$$

satisfy

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!} = e^{2xz-z^2}.$$

Proof: The key idea with which to begin is to consider instead

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2}.$$

We consider the Taylor series expansion of this guy, with respect to z, viewing x as a parameter. By definition, the Taylor series expansion for

$$e^{-(x-z)^2} = \sum_{n\geq 0} a_n z^n,$$

where

$$a_n = \frac{1}{n!}\frac{d^n}{dz^n}e^{-(x-z)^2}, \quad \text{evaluated at } z = 0.$$

To compute these coefficients, we use the chain rule, introducing a new variable $u = x - z$. Then,

$$\frac{d}{dz}e^{-(x-z)^2} = -\frac{d}{du}e^{-u^2},$$

and more generally, each time we differentiate, we get a -1 popping out, so

$$\frac{d^n}{dz^n}e^{-(x-z)^2} = (-1)^n\frac{d^n}{du^n}e^{-u^2},$$

Hence, evaluating with $z = 0$, we have

$$a_n = \frac{1}{n!}(-1)^n\frac{d^n}{du^n}e^{-u^2}, \quad \text{evaluated at } u = x.$$

The reason it's evaluated at $u = x$ is because in our original expression we're expanding in a Taylor series around $z = 0$ and $z = 0 \iff u = x$ since $u = x - z$. Now, of course, we have

$$\frac{d^n}{du^n}e^{-u^2}, \quad \text{evaluated at } u = x = \frac{d^n}{dx^n}e^{-x^2}.$$

Hence, we have the Taylor series expansion

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2} = \sum_{n\geq 0}\frac{z^n}{n!}(-1)^n\frac{d^n}{dx^n}e^{-x^2}.$$

Now, we multiply both sides by e^{x^2} to obtain

$$e^{2xz-z^2} = e^{x^2}\sum_{n\geq 0}\frac{z^n}{n!}(-1)^n\frac{d^n}{dx^n}e^{-x^2}.$$

We can bring e^{x^2} inside because everything converges beautifully. Then, we have

$$e^{2xz-z^2} = \sum_{n\geq 0}\frac{z^n}{n!}e^{x^2}(-1)^n\frac{d^n}{dx^n}e^{-x^2}.$$

Voilà! The definition of the Hermite polynomials is staring us straight in the face! Hence, we have computed

$$e^{2xz-z^2} = \sum_{n\geq 0}\frac{z^n}{n!}H_n(x).$$

The Hermite polynomials come from solving PDEs in parabolic shaped regions of $\mathbb{R}^2$.

1.2. **The Laguerre polynomials.** The Laguerre polynomials come from understanding the quantum mechanics of the hydrogen atom. We shall not get into this[1]

Definition 9. *The Laguerre polynomials,*

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x}).$$

We summarize their properties in the following

Theorem 10 (Properties of Laguerre polynomials)**.** *The Laguerre polynomials are an orthogonal basis for $\mathcal{L}^2$ on $(0,\infty)$ with the weight function $x^\alpha e^{-x}$. Their norms squared,*

$$||L_n^\alpha||^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

They satisfy the Laguerre equation

$$[x^{\alpha+1}e^{-x}(L_n^\alpha)']' + nx^\alpha e^{-x}L_n^\alpha = 0.$$

For $x > 0$ and $|z| < 1$,

$$\sum_{n=0}^\infty L_n^\alpha(x)z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}}.$$

1.3. **Answers to the exercises to be done oneself.**

(1) (5.5.5) Solve the problem

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r,\theta,z) : 0 \le r \le b, 0 \le z \le l\}$$

$$u(r,\theta,0) = 0, \quad u(r,\theta,l) = g(r,\theta), \quad u(b,\theta,z) = 0.$$

Answer:

$$u(r,\theta,z) = \sum_{n\ge0}\sum_{k\ge1}(a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta))J_n(\lambda_{nk}r/b)\sinh(\lambda_{nk}z/b),$$

where λ_{nk} is the k^{th} positive zero of J_n, and

$$b_{nk} = \frac{2}{b^2\pi\sinh\lambda_{nk}}\int_{-\pi}^{\pi}\int_0^b g(r,\theta)\frac{J_n(\lambda_{nk}r)}{J_{n+1}(\lambda_{nk})^2}\sin(n\theta)rdrd\theta,$$

and

$$a_{nk} = frac2b^2\pi\sinh\lambda_{nk}\int_{-\pi}^{\pi}\int_0^b g(r,\theta)\frac{J_n(\lambda_{nk}r)}{J_{n+1}(\lambda_{nk})^2}\cos(n\theta)rdrd\theta.$$

(2) (5.2.4) Demonstrate the identity:

$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$

Hint: Use the recurrence formulas. Integrating by parts is a reasonable idea as well.

[1]Alex Jones does get into it: `https://www.youtube.com/watch?v=i91XV07Vsc0`. Check out the Alex Jones Prison Planet `https://www.youtube.com/watch?v=kn_dHspHd8M`. Turns out that Alex Jones's crazy ranting makes for decent death metal vocals. The gay frogs and America first remix are pretty decent too.

(3) (5.2.8) Prove the reduction formula:
$$\int_0^x s^n J_0(s)ds = x^n J_1(x) + (n-1)x^{n-1} J_0(x) - (n-1)^2 \int_0^x s^{n-2} J_0(s)ds.$$
Hint: Integrate by parts, using the facts that $(xJ_1)' = xJ_0$ and $J_0' = -J_1$.

(4) (5.4.2) Expand the function $f(x) = b^2 - x^2$ in a Fourier-Bessel series of the form
$$\sum_{k \geq 1} c_k J_0(\lambda_k x/b),$$
where λ_k is the k^{th} positive zero of the Bessel function J_0. OBS! Remember to integrate with respect to xdx, polar coordinate style.

Answer:
$$8b^2 \sum \frac{J_0(\lambda_k x/b)}{\lambda_k^3 J_1(\lambda_k)}.$$

(5) (6.4.6) Let $f(x) = 1$ for $x > 0$ and 0 for $x < 0$. (Heavyside function basically). Expand f in a series of Hermite polynomials.

Answer:
$$\frac{1}{2} H_0 + \frac{1}{2\sqrt{\pi}} \sum_0^\infty \frac{(-1)^k}{2^{2k}(2k+1)k!} H_{2k+1}.$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).