FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.03.11

Now for what we've all been waiting for: applications to best approximations!

1.1. **Applications to best approximations.**

Here is a typical problem: find the polynomial, $P(x)$, of at most degree 5 which minimizes

$$\int_a^b |f(x) - P(x)|^2 dx.$$

Here you would be explicitly given the function f as well as the interval from a to b. Since it works the same way, it seems wise to show the general principle. Then, you can use this for your particular problems. We know that the Legendre polynomials are an orthogonal basis for $\mathcal{L}^2$ on $(-1, 1)$. Let's first assume

$$a = -1, \quad b = 1.$$

Then, we compute

$$c_n = \frac{\int_{-1}^1 f(x) P_n(x) dx}{||P_n||^2}, \quad n = 0, 1, 2, 3, 4, 5.$$

The polynomial is, by the best approximation theorem(s),

$$P(x) = \sum_{n=0}^5 c_n P_n(x).$$

So, suppose we don't have $a = -1$ and $b = 1$, but instead we've got some other interval. Let

$$m = \frac{a+b}{2}.$$

This is the midpoint of the interval. Let

$$\ell = \frac{b-a}{2}.$$

Then the interval

$$(a, b) = (m - \ell, m + \ell).$$

So, if we want to move this interval to $(-1,1)$, we take $t \in (m-\ell, m+\ell)$ and map it to

$$t \mapsto \frac{t-m}{\ell} = x.$$

We see that $m \mapsto 0$ and the endpoints

$$m - \ell \mapsto \frac{m-\ell-m}{\ell} = -1, \quad m+\ell \mapsto \frac{m+\ell-m}{\ell} = 1.$$

It is a linear map, so everything in between maps to everything in between -1 and 1. So we have a bijection between (a,b) and $(-1,1)$. If we want to go from $(-1,1)$ to (a,b) then we send

$$x \in (-1,1) \mapsto \ell x + m = t.$$

Since we know about the Legendre polynomials, P_n, on $(-1,1)$ since $t \mapsto \frac{t-m}{\ell} = x$ sends (a,b) to $(-1,1)$,

$$P_n\left(\frac{t-m}{\ell}\right) \qquad \text{are orthogonal on } (a,b).$$

To see this, just compute

$$\int_a^b P_n\left(\frac{t-m}{\ell}\right) P_k\left(\frac{t-m}{\ell}\right) dt = \int_{-1}^1 \ell P_n(x)P_k(x)dx = 0 \text{ if } n \neq k.$$

We have simply used substitution in the integral with $x = \frac{t-m}{\ell}$. So, these modified Legendre polynomials are orthogonal on (a,b). Moreover

$$\int_a^b P_n^2\left(\frac{t-m}{\ell}\right) dt = \int_{-1}^1 \ell P_n^2(x)dx = \ell ||P_n||^2 = \frac{2\ell}{2n+1}.$$

So, we simply expand the function f using this version of the Legendre polynomials. Let

$$c_n = \frac{\int_a^b f(t) P_n\left(\frac{t-m}{\ell}\right) dt}{\int_a^b [P_n((t-m)/\ell)]^2 dt}.$$

The polynomial we seek is

$$P(t) = \sum_{n=0}^5 c_n P_n\left(\frac{t-m}{\ell}\right).$$

1.1.1. *Weighted $\mathcal{L}^2$ on $\mathbb{R}$ with weight e^{-x^2}.* Find the polynomial of at most degree 4 which minimizes

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-x^2} dx.$$

We know that the Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with the weight function e^{-x^2}. We see that same weight function in the integral. Therefore, we can rely on the theory of the Hermite polynomials! Consequently, we define

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(x) e^{-x^2} dx}{||H_n||^2},$$

where

$$||H_n||^2 = \int_{\mathbb{R}} H_n^2(x) e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{4} c_n H_n(x).$$

Some variations on this theme are created by changing the weight function. For example, consider the problem: find the polynomial of at most degree 6 which minimizes

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-2x^2} dx.$$

This is not the correct weight function for H_n. However, we can make it so. The correct weight function for $H_n(x)$ is e^{-x^2}. So, if the exponential has $2x^2 = (\sqrt{2}x)^2$, then we should change the variable in H_n as well. We will then have, via the substitution $t = \sqrt{2}x$,

$$\int_{\mathbb{R}} H_n(\sqrt{2}x) H_m(\sqrt{2}x) e^{-2x^2} dx = \int_{\mathbb{R}} H_n(t) H_m(t) e^{-t^2} \frac{dt}{\sqrt{2}} = 0, \quad n \neq m.$$

Moreover, the norm squared is now

$$\int_{\mathbb{R}} H_n^2(t) e^{-t^2} \frac{dt}{\sqrt{2}} = \frac{||H_n||^2}{\sqrt{2}} = \frac{2^n n! \sqrt{\pi}}{\sqrt{2}}.$$

Consequently, the function $H_n(\sqrt{2}x)$ are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-2x^2}. We have computed the norms squared above. The coefficients are therefore

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(\sqrt{2}x) e^{-2x^2} dx}{2^n n! \sqrt{\pi}/\sqrt{2}}.$$

The polynomial is

$$P(x) = \sum_{n=0}^{6} c_n H_n(\sqrt{2}x).$$

1.1.2. *Weighted $\mathcal{L}^2$ on $(0, \infty)$ with weight $x^\alpha e^{-x}$.* This is rather unlikely to occur, because the Laguerre polynomials are rather scary, but it is possible. So, best that you are prepared for this eventuality. In this case, we know that the Laguerre polynomials are an orthogonal basis for this Hilbert space. So, if we are asked, for example, find the polynomial of at most degree 7 which minimizes

$$\int_0^\infty |f(x) - P(x)|^2 x^\alpha e^{-x} dx,$$

then we should define

$$c_n = \frac{\int_0^\infty f(x) L_n^\alpha(x) x^\alpha e^{-x} dx}{||L_n^\alpha||^2}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{7} c_n L_n^\alpha(x).$$

Variations on this theme? That is virtually unimaginable.

1.1.3. *Other functions and considerations.* We could ask the same type of question looking for coefficients of $\sin(nx)$ or $\cos(nx)$, for say $n = 0, 1, 2, 3, .., N$. Here, one uses the fact that those functions also yield orthogonal basis for $\mathcal{L}^2$ on bounded intervals. That is the name of the game: using the first N elements of an orthogonal basis for the $\mathcal{L}^2$ space under consideration.

You may wonder why when it says *at most degree* N we always find *all* the coefficients, $c_0, c_1, \ldots c_N$. That is because this is *better* then stopping at say c_{N-1}. Find them all. It could turn out that some of these end up being zero, so the polynomial has degree lower than N. The only way to know that is to check the calculation of all the c's, OR to know that certain coefficients will vanish due to evenness or oddness of functions, things of that nature. So, don't toss out any of the coefficients unless you are sure they vanish. Collect them all, like Pokemon!

1.2. **When to use what method?** This course could be renamed to "an introduction to geometric analysis," because in fact, it is. In geometric analysis, we do analysis in different geometric settings. We need to understand how the geometric setting and analysis interact. A real world example of this is a sound check at a heavy metal concert. The particular geometric features of the concert venue and the crowd will affect the way the band sounds. It is quite subtle and difficult, so there is not really a "one setting fits all" solution. That's why a band who has played hundreds of concerts still needs to do a sound check at every concert.

We are fortunately not dealing with a problem as difficult as solving the wave equation in a concert venue. Nonetheless, we still have pretty difficult problems. The way we try to solve them is initially by trying different methods and seeing what works, and what does not. Then, we try to collect the problems into different general types, and explain to students which method will have a reasonable chance of success for which types of problems. However, this is not an exact science. We will only know for sure if we try and see what happens... Much like trying the different sound configurations as we set up our metal concert.

To gain some intuition about Fourier transform methods, the following two theorems are useful.

Theorem 1 (Heat propagates with infinite speed)**.** *Let $f \in \mathcal{L}^2(\mathbb{R})$, and assume f is also continuous. Assume that $f(x) \geq 0$ for all $x \in \mathbb{R}$ and that for some $a \in \mathbb{R}$ we have $f(a) > 0$. Let $u(x,t)$ be the solution to the homogeneous heat equation with initial data given by f. Then for every $x \in \mathbb{R}$ and for every $t > 0$ we have $u(x,t) > 0$.*

Proof: We use the heat kernel to obtain the solution to the heat equation is

$$u(x,t) = \int_{\mathbb{R}} f(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy.$$

By the continuity of f there exists $\delta > 0$ such that

$$|y - a| < \delta \implies f(y) > \frac{f(a)}{2}.$$

Then since $f(y) \geq 0$ for all $y \in \mathbb{R}$ and we also have

$$e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} > 0 \quad \forall y \in \mathbb{R},$$

we obtain the estimate

$$u(x,t) \geq \int_{a-\delta}^{a+\delta} \frac{f(a)}{2} e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy > 0.$$

This shows that as long as our initial data has some heat, somewhere, at time $t = 0$, then for every time positive, that heat has spread across the entire real line. The only way this is possible is if the heat travels infinitely fast. Pretty cool.

Theorem 2 (Fourier transform spreads like peanut butter)**.** *Let $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$. Assume that f has compact support. Then $\hat{f}(\xi)$ has compact support if and only if $f \equiv 0$.*

Proof: By the Lebesgue dominated convergence theorem from measure theory (apologies that we cannot cover that here...) the function $\hat{f}(\xi)$ is entire. That means it is holomorphic (synonym for that is analytic) in the entire complex plane. Assume that $\hat{f}(\xi)$ restricted to the real axis has compact support. This means that $\hat{f}(\xi) = 0$ for all real ξ outside of a bounded interval. By the identity theorem from complex analysis, this means that the function $\hat{f} \equiv 0$. By the Fourier Inversion Theorem, that in turn means that $f \equiv 0$.

This shows that the Fourier transform smears the values of a function over the entire real line. For example, let us compute the Fourier transform of the function which is equal to one on the interval $[0,1]$ and zero everywhere else:

$$\hat{f}(\xi) = \int_0^1 e^{-ix\xi} dx = \begin{cases} -\frac{e^{-i\xi}}{i\xi} + \frac{1}{i\xi} & \xi \neq 0 \\ 1 & \xi = 0 \end{cases}.$$

This simplifies to

$$\hat{f}(\xi) = \begin{cases} \frac{i}{\xi}\left(\cos(\xi) - 1\right) + \frac{\sin(\xi)}{\xi} & \xi \neq 0 \\ 1 & \xi = 0 \end{cases}.$$

When is this function zero? It is zero precisely when

$$\xi = 2k\pi, \quad k \in \mathbb{Z} \setminus \{0\}.$$

For all other values of ξ the function $\hat{f}(\xi)$ is *not* zero. This shows that there is no bounded interval such that $\hat{f}$ vanishes outside that interval. Indeed even though f is zero outside the interval $[0,1]$ the same is simply not true for $\hat{f}$. The Fourier transform has smeared the positive values of f within the interval $[0,1]$ onto the real axis. Like peanut butter.

1.2.1. *Applications to PDEs.* To solve a PDE on the entire real line, a good technique to try is the Fourier transform. We have two examples of this: the homogeneous heat equation and the inhomogeneous heat equation. For the homogeneous heat equation with initial data $u(x,0) = v(x)$ the solution is

$$u(x,t) = \int_{\mathbb{R}} v(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy.$$

For the inhomogeneous heat equation with the same initial data

$$\partial_t U(x,t) - \partial_{xx} U(x,t) = G(x,t)$$

the Fourier transform method leads to the solution

$$U(x,t) = \int_{\mathbb{R}} v(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy + \int_0^t \int_{\mathbb{R}} G(y,s) e^{-(x-y)^2/(4(t-s))} (4\pi(t-s))^{-1/2} dy ds.$$

Let us now consider the initial value problem for the homogeneous wave equation:

$$\begin{cases} u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \\ \Box u = 0 & x \in \mathbb{R},\, t > 0 \end{cases}.$$

We hit the PDE with the Fourier transform in the x variable:

$$\hat{u}_{tt}(\xi,t) - \widehat{u_{xx}}(\xi,t) = 0.$$

We use the fact that the Fourier transform turns the x derivatives into multiplication by $i\xi$ to obtain:

$$\hat{u}_{tt}(\xi,t) + \xi^2 \hat{u}(\xi,t) = 0.$$

We re-arrange to see:

$$\hat{u}_{tt}(\xi,t) = -\xi^2 \hat{u}(\xi,t).$$

Consequently

$$\hat{u}(\xi,t) = a(\xi)\cos(\xi t) + b(\xi)\sin(\xi t).$$

To determine the coefficient functions we use the IC. First

$$\hat{u}(\xi,0) = a(\xi) = \hat{f}(\xi).$$

Second,

$$\hat{u}_t(\xi,0) = \xi b(\xi) = \hat{g}(\xi) \implies b(\xi) = \frac{\hat{g}(\xi)}{\xi}.$$

So, we have obtained

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos(\xi t) + \hat{g}(\xi)\frac{\sin(\xi t)}{\xi}.$$

We see from our handy table that Fourier transform of $\frac{1}{2}\chi_t(x)$ is equal to $\frac{\sin(\xi t)}{\xi}$. Here the function

$$\chi_t(x) = \begin{cases} 1 & |x| \le t \\ 0 & |x| > t \end{cases}.$$

I am a little worried about the cosine term because the cosine is very much not an element of $\mathcal{L}^2$. However, we can write it using complex exponentials and then use properties of the Fourier transform.

$$\hat{f}(\xi)\cos(\xi t) = \frac{1}{2}\left(\hat{f}(\xi)e^{i\xi t} + \hat{f}(\xi)e^{-i\xi t}\right).$$

The handy table of properties of the Fourier transform says that the Fourier transform of $f(x \pm t)$ is $\hat{f}(\xi)e^{\pm i\xi t}$. So going backwards, our solution

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{\mathbb{R}} g(y)\chi_t(x-y)dy.$$

If we feel like it, we can simplify the second term a bit,

$$\frac{1}{2}\int_{\mathbb{R}} g(y)\chi_t(x-y)dy = \frac{1}{2}\int_{x-t}^{x+t} g(y)dy,$$

so that

$$u(x,t) = \frac{1}{2}\left(f(x+t)+f(x-t)+\int_{x-t}^{x+t} g(y)dy\right).$$

1.3. **Legendre polynomials origins story.** We consider spherical coordinates in $\mathbb{R}^3$. These coordinates are useful for solving PDEs inside spheres or pieces of spheres. The spherical coordinates are (r, θ, ϕ). The first coordinate, r tells us the distance of the point in $\mathbb{R}^3$ to the origin. The second coordinate, θ, tells us the angle of the point in the $x-y$ plane. The third coordinate, ϕ, tells the angle of the point in the z direction. So, if $\phi = 0$, the point is along the positive z-axis. If $\phi = \frac{\pi}{2}$, the point has z-coordinate equal to zero. If $\phi = \pi$, the point is along the negative z-axis. The standard coordinate are therefore

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi.$$

To see how this work, draw some right triangles from different perspectives (will do in lecture!). By the chain rule, the Laplace operator

$$\Delta = -\partial_x^2 - \partial_y^2 - \partial_z^2 = -\partial_r^2 - \frac{2}{r}\partial_r - \frac{\sin\phi\partial_\phi^2 + \cos\phi\partial_\phi)}{r^2\sin\phi} - \frac{\partial_\theta^2}{r^2\sin^2\phi}.$$

Consider solving the Dirichlet problem inside a sphere. We would like $\Delta u = 0$. Since the natural coordinates on a sphere are the spherical coordinates, we write u as a product of three functions depending on the three spherical coordinates,

$$R(r)\Theta(\theta)\Phi(\phi).$$

Then, the PDE becomes

$$\Delta(R\Theta\Phi) = 0 \implies \frac{R''}{R} + \frac{2R'}{rR} + \frac{\Phi''\sin\phi + \Phi'\cos\phi}{r^2\sin\phi\Phi} + \frac{\Theta''}{r^2\sin^2\phi\Theta} = 0.$$

Let us use φ for the variable, ϕ, and continue to use Φ for the function. We multiply by $r^2\sin^2\varphi$:

$$\frac{R''r^2\sin^2\varphi}{R} + \frac{r\sin^2\varphi 2R'}{R} + \frac{\sin\varphi(\Phi''\sin\varphi + \Phi'\cos\varphi}{\Phi} + \frac{\Theta''}{\Theta} = 0.$$

Since it is the most simple, we move Θ to the other side:

$$\frac{R''r^2\sin^2\varphi}{R} + \frac{r\sin^2\varphi 2R'}{R} + \frac{\sin\varphi(\Phi''\sin\varphi + \Phi'\cos\varphi}{\Phi} = -\frac{\Theta''}{\Theta}.$$

Therefore both sides are constant. We deal with Θ first. Conquer the weakest opponents first, so that they are not trying to attack from behind whilst one deals with the more significant threats. The equation for Θ is by far the simplest. For geometric reasons, Θ must be a 2π periodic function. Therefore

$$-\frac{\Theta''}{\Theta} = m^2, \quad m \in \mathbb{Z}, \quad \Theta_m(\theta) = e^{im\theta}.$$

We therefore can use this in the equation for the right side:

$$\frac{R''r^2\sin^2\varphi}{R} + \frac{r\sin^2\varphi 2R'}{R} + \frac{\sin\varphi(\Phi''\sin\varphi + \Phi'\cos\varphi}{\Phi} = m^2.$$

We divide by $\sin^2\varphi$ and move all the φ dependent terms to the right side, obtaining

$$\frac{R''r^2+2rR'}{R}=\frac{m^2}{\sin^2\varphi}-\left(\frac{\sin\varphi\Phi''+\cos\varphi\Phi'}{\sin\varphi\Phi}\right).$$

Similarly, as both sides depend on different variables, both sides must be constant. So, we shall call the constant λ. We shall deal with the φ business first, doing a clever transformation. Let

$$s=\cos\varphi.$$

Then we note that $\cos:[0,\pi]\to[-1,1]$ bijectively. We also have $\varphi=\arccos s$. Let

$$S(s):=S(\cos\varphi)=\Phi(\varphi).$$

Then by the chain rule,

$$\Phi'(\varphi)=-\sin\varphi S'(s),\quad \Phi''(\varphi)=-\cos\varphi S'(s)+\sin^2\varphi S''(s).$$

By definition of s, and the fact that $\sin^2+\cos^2=1$,

$$\Phi''(\varphi)=-sS'(s)+(1-s^2)S''(s).$$

We therefore see that

$$\frac{\Phi''}{\Phi}=\frac{-sS'+(1-s^2)S''}{\Phi},\quad \frac{\Phi'\cos\varphi}{\Phi\sin\varphi}=\frac{-\sin\varphi\cos\varphi S'}{\sin\varphi S}=-\frac{sS'}{S}.$$

The equation for the φ variable side is then

$$\lambda=\frac{m^2}{1-s^2}-\left(\frac{-sS'+(1-s^2)S''}{S}-\frac{sS'}{S}\right)=\lambda.$$

We multiply by S and obtain

$$\frac{Sm^2}{1-s^2}-\left(-2sS'+(1-s^2)S''\right)=\lambda S.$$

Observe that

$$-2sS'+(1-s^2)S''=\left[(1-s^2)S'\right]'.$$

So, the equation is

legm
$$\frac{Sm^2}{1-s^2}-\left[(1-s^2)S'\right]'-\lambda S=0. \tag{1.1}$$

If $m=0$, this equation is

leg0
$$-\left[(1-s^2)S'\right]'-\lambda S=0 \iff \left[(1-s^2)S'\right]'+\lambda S=0. \tag{1.2}$$

Since $m\in\mathbb{Z}$, we would like to find solutions to this equation. The easiest case is the case when $m=0$. It turns out that the Legendre polynomials solve this equation.

Theorem 3. *The Legendre polynomials solve*

$$\left[(1-x^2)P_n'(x)\right]'+n(n+1)P_n(x)=0.$$

In particular, they are eigenfunctions for the SLP $[(1-x^2)'u']'+\lambda u=0$ *with eigenvalues* $\lambda=n(n+1)$.

Proof: By the product rule,

$$[(1-x^2)P_n']' = -2xP_n' + (1-x^2)P_n''.$$

We compute the leading coefficient coming from

$$-2xP_n' - x^2P_n''.$$

We recall that

$$P_n(x) = \frac{1}{2^n n!}\sum_{k\geq n/2}^{n}(-1)^{n-k}\binom{n}{k}x^{2k-n}\prod_{j=0}^{n-1}(2k-j).$$

The highest order term comes from $k=n$, and it is

$$\frac{1}{2^n n!}x^n\prod_{j=0}^{n-1}(2n-j) = \frac{1}{2^n n!}x^n\frac{(2n)!}{n!}.$$

We therefore compute that

$$-2xP_n' - x^2P_n'' = -\frac{2n(2n)!x^n}{2^n(n!)^2} - \frac{n(n-1)(2n)!x^n}{2^n(n!)^2} = \frac{(2n)!x^n(-2n-n(n-1))}{2^n(n!)^2}$$
$$= -\frac{(2n)!x^n n(n+1)}{2^n(n!)^2}.$$

If we look back at the highest order term in P_n itself, this was

$$\frac{(2n)!x^n}{2^n(n!)^2}.$$

So we see that the highest order term in

$$[(1-x^2)P_n']' \text{ is } -n(n+1)\frac{(2n)!x^n}{2^n(n!)^2}.$$

Consequently

$$[(1-x^2)P_n']' + n(n-1)P_n \text{ is a polynomial of degree } n-1 \text{ or lower.}$$

We may therefore express this polynomial, call it q as a linear combination of the Legendre polynomials of degree up to $n-1$, that is

$$q = \sum_{j=0}^{n-1}c_jP_j.$$

Let us compute the coefficients:

$$c_j = \frac{\langle q, P_j\rangle}{||P_j||^2}.$$

We first compute using integration by parts and the vanishing of the boundary terms:

$$\int_{-1}^{1}[(1-x^2)P_n']'P_jdx = -\int_{-1}^{1}(1-x^2)P_n'P_j'dx = \int_{-1}^{1}[(1-x^2)P_j']'P_ndx.$$

Observe that $[(1-x^2)P_j']'$ is a polynomial of degree $j<n$. It can therefore be written as a linear combination of $P_0,\ldots,P_j$. Each of these are orthogonal to P_n. Hence this part vanishes. For the second part, we compute

$$\int_{-1}^{1}n(n+1)P_n(x)P_j(x)dx = 0,$$

since $j < n$. So in fact all together, $c_j = 0$ for all $j = 0, \ldots, n-1$. We therefore have computed that

$$[(1-x^2)P_n']' + n(n-1)P_n = 0.$$

For $m = 0$, the functions $P_n(s)$ solves the equation (1.2), with $\lambda_n = n(n+1)$. For the general case, I leave it as an exercise to verify that

$$P_n^m(s) := (1-s^2)^{|m|/2} \frac{d^{|m|}}{ds^{|m|}} P_n(s)$$

solves (1.2). Recalling that $s = \cos\varphi$, we have therefore found functions

$$\Theta_m(\theta) = e^{im\theta},$$

and

$$P_n^m(\varphi) = (1-s^2)^{|m|/2} \frac{d^{|m|}}{ds^{|m|}} P_n(s) \text{ first compute the derivative, then set } s = \cos\varphi.$$

Finally, we use the value of $\lambda = n(n+1)$ to solve for the function R:

$$\frac{R''r^2 + 2rR'}{R} = \lambda_n = n(n+1).$$

This becomes

$$R''r^2 + 2rR' - \lambda_n R = 0.$$

This is an Euler equation. We look for solutions of the form $R(r) = r^\alpha$. Putting such a function into the ODE,

$$\alpha(\alpha-1)r^\alpha + 2\alpha r^\alpha - \lambda_n r^\alpha = 0 \iff \alpha^2 + \alpha - \lambda_n = 0.$$

We solve the quadratic equation for

$$\alpha = \frac{-1 \pm \sqrt{1+4\lambda_n}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1+4n(n+1)}}{2}.$$

We do not want $R(r) \to \infty$ when $r \to 0$, so we choose the solution with the plus. We fiddle a little with this square root part:

$$\frac{\sqrt{1+4n(n+1)}}{2} = \sqrt{\frac{1}{4} + n(n+1)} = \sqrt{n^2 + n + \frac{1}{4}} = \sqrt{(n+1/2)^2} = n + 1/2.$$

Consequently

$$-\frac{1}{2} + \frac{\sqrt{1+4n(n+1)}}{2} = n.$$

We have therefore found

$$R_n(r) = r^n.$$

Up to constant factors, we have thus found the functions

$$u_{m,n}(r,\theta,\varphi) = r^n e^{im\theta} P_n^m(\cos\varphi),$$

which solve

$$\Delta u_{m,n} = 0$$

in the sphere. It just so happens that we can smash them all together and solve the Dirichlet problem in a sphere.

Theorem 4. *The solution to the Dirichlet problem in the unit sphere in* $\mathbb{R}^3$*, that is*

$$\Delta u = 0, \quad u(1, \theta, \varphi) = f(\theta, \varphi)$$

is

$$u(r, \theta, \varphi) = \sum_{n \geq 0, m \in \mathbb{Z}} \widehat{f_{n,m}} r^n e^{im\theta} P_n^m(\cos \varphi),$$

with

$$\widehat{f_{n,m}} = \frac{\int_0^\pi \int_0^{2\pi} f(\theta, \varphi) e^{-im\theta} P_n^m(\cos \varphi) d\theta \sin \varphi d\varphi}{2\pi ||P_n^m||^2} = \frac{\int_{-1}^1 \int_0^{2\pi} f(\theta, \arccos(s)) e^{-im\theta} P_n^m(s) d\theta ds}{2\pi ||P_n^m||^2}.$$

The functions

$$Y_{m,n}(\theta, \varphi) = e^{im\theta} P_n^m(\varphi)$$

are called *spherical harmonics.* One can show that

$$||P_n^m||^2 = \frac{(n+m)!2}{(n-m)!(2n+1)}, \quad n \geq |m|,$$

and that

$$||P_n^m||^2 = 0, \quad n < |m|.$$

We have deserved some comic relief. This shall be provided by the French song, Foux du Fa Fa, an exerpt from the series, Flight of the Conchords `https://www.youtube.com/watch?v=EuXdhow3uqQ`. Parlez-vous le français?

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).